Planarity via Spanning Tree Number: A Linear-Algebraic Criterion

Alan Bu[1] **Yuchong Pan[2]**

[1]Harvard University **[2]Massachusetts Institute of Technology**

May 23, 2024
Algorithms Seminar
University of British Columbia

planarity criteria

Planarity criteria

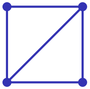
Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Planarity criteria

Definition

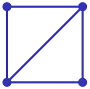
A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

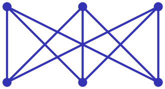


Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

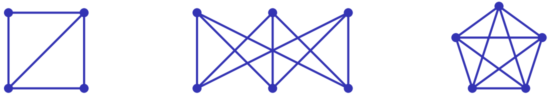




Planarity criteria

Definition

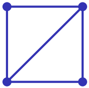
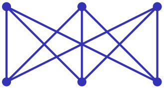
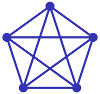
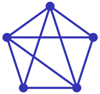
A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.



Planarity criteria

Definition

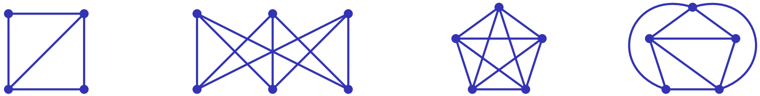
A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

Planarity criteria

Definition

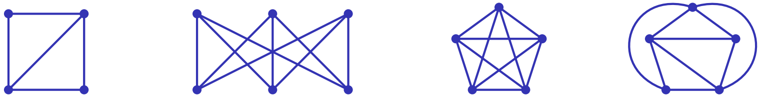
A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.



Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

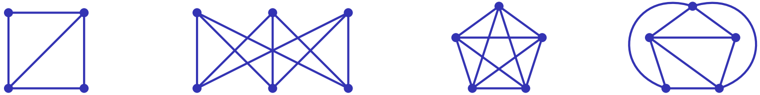


1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$

Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

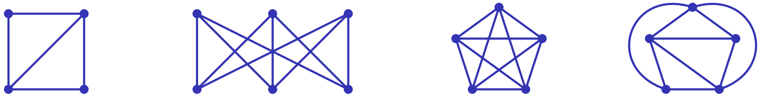


1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$
2. Whitney (1932): planar iff. its graphic matroid is cographic

Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

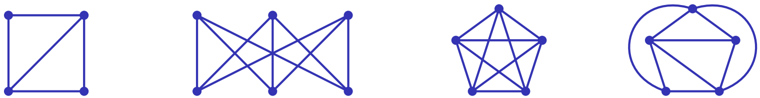


1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$
2. Whitney (1932): planar iff. its graphic matroid is cographic
3. Wagner (1937): planar iff. its minors do not include K_5 or $K_{3,3}$

Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

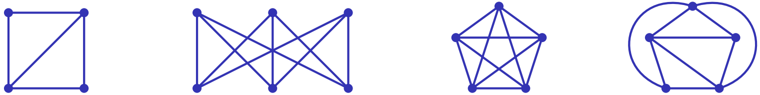


1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$
2. Whitney (1932): planar iff. its graphic matroid is cographic
3. Wagner (1937): planar iff. its minors do not include K_5 or $K_{3,3}$
4. Mac Lane (1937): planar iff. its cycle space has a 2-basis

Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

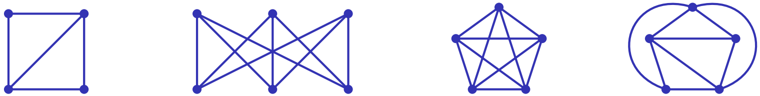


1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$
2. Whitney (1932): planar iff. its graphic matroid is cographic
3. Wagner (1937): planar iff. its minors do not include K_5 or $K_{3,3}$
4. Mac Lane (1937): planar iff. its cycle space has a 2-basis
5. de Fraysseix–Rosenstiehl (1982, 1985): criterion based on depth-first search trees

Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

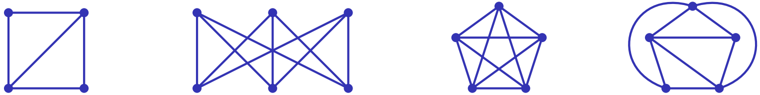


1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$
2. Whitney (1932): planar iff. its graphic matroid is cographic
3. Wagner (1937): planar iff. its minors do not include K_5 or $K_{3,3}$
4. Mac Lane (1937): planar iff. its cycle space has a 2-basis
5. de Fraysseix–Rosenstiehl (1982, 1985): criterion based on depth-first search trees
6. Hanani-Tutte (1934, 1970): criterion based on the parity of edge crossings

Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.

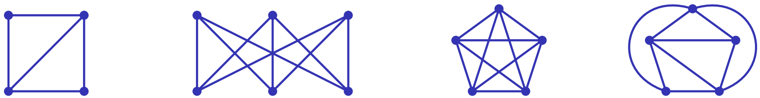


1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$
2. Whitney (1932): planar iff. its graphic matroid is cographic
3. Wagner (1937): planar iff. its minors do not include K_5 or $K_{3,3}$
4. Mac Lane (1937): planar iff. its cycle space has a 2-basis
5. de Fraysseix–Rosenstiehl (1982, 1985): criterion based on depth-first search trees
6. Hanani-Tutte (1934, 1970): criterion based on the parity of edge crossings
7. Schnyder (1989): criterion based on poset dimensions

Planarity criteria

Definition

A graph is planar if it can be drawn on the plane in such a way that its edges intersect only at their endpoints.



1. Kuratowski (1930): planar iff. it does not contain a subdivision of K_5 or $K_{3,3}$
2. Whitney (1932): planar iff. its graphic matroid is cographic
3. Wagner (1937): planar iff. its minors do not include K_5 or $K_{3,3}$
4. Mac Lane (1937): planar iff. its cycle space has a 2-basis
5. de Fraysseix–Rosenstiehl (1982, 1985): criterion based on depth-first search trees
6. Hanani-Tutte (1934, 1970): criterion based on the parity of edge crossings
7. Schnyder (1989): criterion based on poset dimensions
8. ...

We give a linear-algebraic planarity criterion based on the number of spanning trees.

Incidence matrices and variants

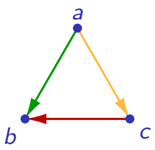
Definition

Given a graph G with orientation $D = (V, A)$, its incidence matrix is an $A \times V$ matrix, denoted by $\iota_D = (a_{e,v})$, $a_{e,v}$ is 1 if e enters v, -1 if e leaves v, and 0 otherwise.

Incidence matrices and variants

Definition

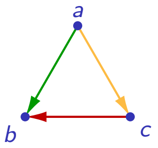
Given a graph G with orientation $D = (V, A)$, its incidence matrix is an $A \times V$ matrix, denoted by $\iota_D = (a_{e,v})$, $a_{e,v}$ is 1 if e enters v, -1 if e leaves v, and 0 otherwise.



Incidence matrices and variants

Definition

Given a graph G with orientation $D = (V, A)$, its incidence matrix is an $A \times V$ matrix, denoted by $\iota_D = (a_{e,v})$, $a_{e,v}$ is 1 if e enters v, -1 if e leaves v, and 0 otherwise.

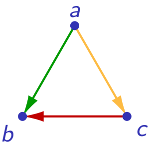


$$\begin{array}{c} \begin{matrix} a & b & c \end{matrix} \\ \begin{bmatrix} -1 & 1 & \\ -1 & & 1 \\ & 1 & -1 \end{bmatrix} \end{array}$$

Incidence matrices and variants

Definition

Given a graph G with orientation $D = (V, A)$, its incidence matrix is an $A \times V$ matrix, denoted by $\iota_D = (a_{e,v})$, $a_{e,v}$ is 1 if e enters v, -1 if e leaves v, and 0 otherwise.



$$\begin{array}{ccc} a & b & c \end{array}$$
$$\begin{bmatrix} -1 & 1 & \\ -1 & & 1 \\ & 1 & -1 \end{bmatrix}$$

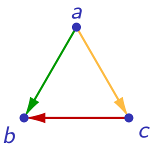
Definition

A truncated incidence matrix of a graph (resp. digraph), denoted by $\mathrm{trun}(\cdot)$, is an incidence matrix of the graph (resp. digraph) with an arbitrary column removed.

Incidence matrices and variants

Definition

Given a graph G with orientation $D = (V, A)$, its incidence matrix is an $A \times V$ matrix, denoted by $\iota_D = (a_{e,v})$, $a_{e,v}$ is 1 if e enters v, -1 if e leaves v, and 0 otherwise.



$$\begin{array}{ccc} a & b & c \end{array}$$
$$\begin{bmatrix} -1 & 1 & \\ -1 & & 1 \\ & 1 & -1 \end{bmatrix}$$

Definition

A truncated incidence matrix of a graph (resp. digraph), denoted by $\mathrm{trun}(\cdot)$, is an incidence matrix of the graph (resp. digraph) with an arbitrary column removed.

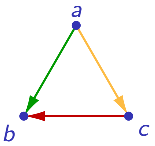
Definition

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

Incidence matrices and variants

Definition

Given a graph G with orientation $D = (V, A)$, its incidence matrix is an $A \times V$ matrix, denoted by $\iota_D = (a_{e,v})$, $a_{e,v}$ is 1 if e enters v, -1 if e leaves v, and 0 otherwise.



$$\begin{array}{ccc} a & b & c \end{array}$$
$$\begin{bmatrix} -1 & 1 & \\ -1 & & 1 \\ & 1 & -1 \end{bmatrix}$$

Definition

A truncated incidence matrix of a graph (resp. digraph), denoted by $\mathrm{trun}(\cdot)$, is an incidence matrix of the graph (resp. digraph) with an arbitrary column removed.

Definition

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero. A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Our criterion

Our criterion

Definition

Given a connected graph G,

- we use maxdet(G) to denote the maximum determinant of a square concatenation $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size;

Our criterion

Definition

Given a connected graph G,

- we use $\text{maxdet}(G)$ to denote the maximum determinant of a square concatenation $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size;
- we define its excess to be $\varepsilon(G) := \tau(G) - \text{maxdet}(G)$.

Our criterion

Definition

Given a connected graph G,

- we use maxdet(G) to denote the maximum determinant of a square concatenation $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size;
- we define its excess to be $\varepsilon(G) := \tau(G) - \text{maxdet}(G)$.

Given a disconnected graph G, we define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

Our criterion

Definition

Given a connected graph G,

- we use maxdet(G) to denote the maximum determinant of a square concatenation $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size;
- we define its excess to be $\varepsilon(G) := \tau(G) - \operatorname{maxdet}(G)$.

Given a disconnected graph G, we define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

Theorem (Bu, **P.** '2024+)

For any graph G,

$$\begin{aligned} \varepsilon(G) &= 0, && \text{if } G \text{ is planar,} \\ \varepsilon(G) &\geq 18, && \text{if } G \text{ is nonplanar.} \end{aligned}$$

Our criterion

Definition

Given a connected graph G,

- we use maxdet(G) to denote the maximum determinant of a square concatenation $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size;
- we define its excess to be $\varepsilon(G) := \tau(G) - \text{maxdet}(G)$.

Given a disconnected graph G, we define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

Theorem (Bu, **P.** '2024+)

For any graph G,

$$\begin{aligned} \varepsilon(G) &= 0, && \text{if } G \text{ is planar,} \\ \varepsilon(G) &\geq 18, && \text{if } G \text{ is nonplanar.} \end{aligned}$$

- This gives a certificate of planarity of a planar graph that can be verified by computing a determinant and counting spanning trees.

Our criterion

Definition

Given a connected graph G,

- we use $\mathrm{maxdet}(G)$ to denote the maximum determinant of a square concatenation $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size;
- we define its excess to be $\varepsilon(G) := \tau(G) - \mathrm{maxdet}(G)$.

Given a disconnected graph G, we define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

Theorem (Bu, **P.** '2024+)

For any graph G,

$$\begin{aligned} \varepsilon(G) &= 0, && \text{if } G \text{ is planar,} \\ \varepsilon(G) &\geq 18, && \text{if } G \text{ is nonplanar.} \end{aligned}$$

- This gives a certificate of planarity of a planar graph that can be verified by computing a determinant and counting spanning trees.
- $\varepsilon(K_{3,3}) = 18$.

Case of planar graphs

Lemma

Let G be a connected planar graph with orientation D.

Case of planar graphs

Lemma

Let G be a connected planar graph with orientation D. Let D^* be the planar dual of D.

Case of planar graphs

Lemma

Let G be a connected planar graph with orientation D. Let D^* be the planar dual of D. For each i, suppose that the i^{th} rows of $\mathrm{trun}(D)$ *and* $\mathrm{trun}(D^*)$ *correspond to the same arc (and its dual arc).*

Case of planar graphs

Lemma

Let G be a connected planar graph with orientation D. Let D^* be the planar dual of D. For each i, suppose that the i^{th} rows of $\mathrm{trun}(D)$ and $\mathrm{trun}(D^*)$ correspond to the same arc (and its dual arc). Then

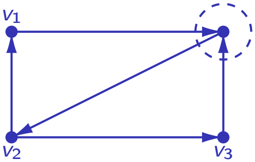
$$\left| \det \left[\mathrm{trun}(D) \mid \mathrm{trun}\left(D^*\right)\right] \right| = \tau(G).$$

Case of planar graphs

Lemma

Let G be a connected planar graph with orientation D. Let D^* be the planar dual of D. For each i, suppose that the i^{th} rows of $\mathrm{trun}(D)$ and $\mathrm{trun}(D^*)$ correspond to the same arc (and its dual arc). Then

$$\left| \det \left[\mathrm{trun}(D) \mid \mathrm{trun}\left(D^*\right) \right] \right| = \tau(G).$$

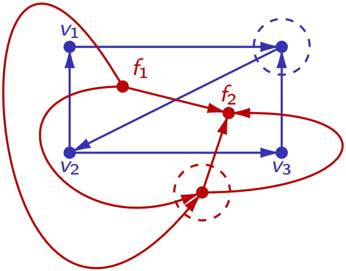


Case of planar graphs

Lemma

Let G be a connected planar graph with orientation D. Let D^* be the planar dual of D. For each i, suppose that the i^{th} rows of $\mathrm{trun}(D)$ and $\mathrm{trun}(D^*)$ correspond to the same arc (and its dual arc). Then

$$\left| \det \left[\mathrm{trun}(D) \mid \mathrm{trun}\left(D^*\right) \right] \right| = \tau(G).$$

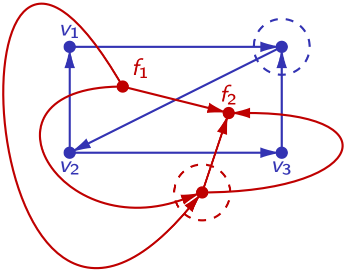


Case of planar graphs

Lemma

Let G be a connected planar graph with orientation D. Let D^* be the planar dual of D. For each i, suppose that the i^{th} rows of $\text{trun}(D)$ and $\text{trun}(D^*)$ correspond to the same arc (and its dual arc). Then

$$\left| \det \left[\text{trun}(D) \mid \text{trun}\left(D^*\right) \right] \right| = \tau(G).$$



v_1	v_2	v_3	f_1	f_2
-1			-1	
1	-1		-1	
	1		-1	1
		-1		1
	-1	1		1

Proof

Proof

$$\left(\det \left[\ \operatorname{trun}(D) \;\middle|\; \operatorname{trun}\left(D^*\right)\ \right]\right)^2 = \det \left[\ \operatorname{trun}(D) \;\middle|\; \operatorname{trun}\left(D^*\right)\ \right]^{\mathsf{T}} \left[\ \operatorname{trun}(D) \;\middle|\; \operatorname{trun}\left(D^*\right)\ \right]$$

$$= \det \begin{bmatrix} \operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D) & \operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}\left(D^*\right) \\ \operatorname{trun}\left(D^*\right)^{\mathsf{T}} \operatorname{trun}(D) & \operatorname{trun}\left(D^*\right)^{\mathsf{T}} \operatorname{trun}\left(D^*\right) \end{bmatrix}.$$

Proof

$$\left(\det \left[\ \operatorname{trun}(D) \mid \operatorname{trun}(D^*)\ \right]\right)^2 = \det \left[\ \operatorname{trun}(D) \mid \operatorname{trun}(D^*)\ \right]^{\mathsf{T}} \left[\ \operatorname{trun}(D) \mid \operatorname{trun}(D^*)\ \right]$$

$$= \det \begin{bmatrix} \operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D) & \operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D^*) \\ \operatorname{trun}(D^*)^{\mathsf{T}} \operatorname{trun}(D) & \operatorname{trun}(D^*)^{\mathsf{T}} \operatorname{trun}(D^*) \end{bmatrix}.$$

Ingredient 1

$\operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D)$ is the Laplacian of G with the row and column indexed by the truncated vertex removed.

Proof

$$\left(\det\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}\left(D^*\right)\ \right]\right)^2 = \det\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}\left(D^*\right)\ \right]^{\mathsf{T}}\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}\left(D^*\right)\ \right]$$
$$= \det\begin{bmatrix} \mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}(D) & \mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}\left(D^*\right) \\ \mathrm{trun}\left(D^*\right)^{\mathsf{T}}\,\mathrm{trun}(D) & \mathrm{trun}\left(D^*\right)^{\mathsf{T}}\,\mathrm{trun}\left(D^*\right) \end{bmatrix}.$$

Ingredient 1

$\mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}(D)$ is the Laplacian of G with the row and column indexed by the truncated vertex removed.

Ingredient 2 (Kirchhoff's matrix-tree theorem)

$\tau(G)$ is equal to the determinant of its Laplacian with the row and column indexed by the same vertex removed.

Proof

$$\left(\det\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}\left(D^*\right)\ \right]\right)^2 = \det\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}\left(D^*\right)\ \right]^{\mathsf{T}}\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}\left(D^*\right)\ \right]$$
$$= \det\begin{bmatrix} \mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}(D) & \mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}\left(D^*\right) \\ \mathrm{trun}\left(D^*\right)^{\mathsf{T}}\,\mathrm{trun}(D) & \mathrm{trun}\left(D^*\right)^{\mathsf{T}}\,\mathrm{trun}\left(D^*\right) \end{bmatrix}.$$

Ingredient 1

$\mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}(D)$ is the Laplacian of G with the row and column indexed by the truncated vertex removed.

Ingredient 2 (Kirchhoff's matrix-tree theorem)

$\tau(G)$ is equal to the determinant of its Laplacian with the row and column indexed by the same vertex removed.

Ingredient 3 (folklore)

If G is a connected planar graph, then $\tau(G^*) = \tau(G)$.

Proof (cont'd)

Hope

$\text{trun}(D)^{\mathsf{T}}\,\text{trun}\,(D^{*}) = 0$,

Proof (cont'd)

Hope

$\mathrm{trun}(D)^\mathsf{T}\,\mathrm{trun}(D^*) = 0$, so

$$
\begin{aligned}
\left(\det\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}(D^*)\ \right]\right)^2 &= \det\begin{bmatrix}\mathrm{trun}(D)^\mathsf{T}\,\mathrm{trun}(D) & 0\\ 0 & \mathrm{trun}(D^*)^\mathsf{T}\,\mathrm{trun}(D^*)\end{bmatrix}\\
&= \det\left(\mathrm{trun}(D)^\mathsf{T}\,\mathrm{trun}(D)\right)\cdot\det\left(\mathrm{trun}(D^*)^\mathsf{T}\,\mathrm{trun}(D^*)\right)\\
&= \tau(G)\cdot\tau(G^*) = \tau(G)^2.
\end{aligned}
$$

Proof (cont'd)

Hope

$\operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D^*) = 0$, so

$$\left(\det \left[\ \operatorname{trun}(D) \mid \operatorname{trun}(D^*)\ \right]\right)^2 = \det \begin{bmatrix} \operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D) & 0 \\ 0 & \operatorname{trun}(D^*)^{\mathsf{T}} \operatorname{trun}(D^*) \end{bmatrix}$$

$$= \det\left(\operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D)\right) \cdot \det\left(\operatorname{trun}(D^*)^{\mathsf{T}} \operatorname{trun}(D^*)\right)$$

$$= \tau(G) \cdot \tau(G^*) = \tau(G)^2.$$

Proof of Hope. Let c be a column of $\operatorname{trun}(D)$ corresponding to vertex v of D. Let c' be a column of $\operatorname{trun}(D^*)$ corresponding to face f of D. Want: $c^{\mathsf{T}} c' = 0$.

Proof (cont'd)

Hope

$\operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D^*) = 0$, so

$$\left(\det \left[\ \operatorname{trun}(D) \mid \operatorname{trun}(D^*)\ \right]\right)^2 = \det \begin{bmatrix} \operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D) & 0 \\ 0 & \operatorname{trun}(D^*)^{\mathsf{T}} \operatorname{trun}(D^*) \end{bmatrix}$$
$$= \det \left(\operatorname{trun}(D)^{\mathsf{T}} \operatorname{trun}(D)\right) \cdot \det \left(\operatorname{trun}(D^*)^{\mathsf{T}} \operatorname{trun}(D^*)\right)$$
$$= \tau(G) \cdot \tau(G^*) = \tau(G)^2.$$

Proof of Hope. Let c be a column of $\operatorname{trun}(D)$ corresponding to vertex v of D. Let c' be a column of $\operatorname{trun}(D^*)$ corresponding to face f of D. Want: $c^{\mathsf{T}} c' = 0$.

If there is no arc incident to both v and f, done.

Proof (cont'd)

Hope

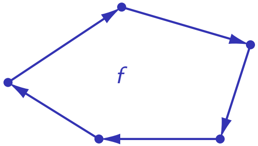
$\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D^*) = 0$, so

$$\left(\det\left[\ \text{trun}(D)\ \middle|\ \text{trun}(D^*)\ \right]\right)^2 = \det\begin{bmatrix}\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D) & 0 \\ 0 & \text{trun}(D^*)^{\mathsf{T}}\,\text{trun}(D^*)\end{bmatrix}$$
$$= \det\left(\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D)\right)\cdot\det\left(\text{trun}(D^*)^{\mathsf{T}}\,\text{trun}(D^*)\right)$$
$$= \tau(G)\cdot\tau(G^*) = \tau(G)^2.$$

Proof of Hope. Let c be a column of $\text{trun}(D)$ corresponding to vertex v of D. Let c' be a column of $\text{trun}(D^*)$ corresponding to face f of D. Want: $c^{\mathsf{T}}c' = 0$.

If there is no arc incident to both v and f, done.

Otherwise,



Proof (cont'd)

Hope

$\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D^*) = 0$, so

$$\left(\det\left[\ \text{trun}(D)\ \middle|\ \text{trun}(D^*)\ \right]\right)^2 = \det\begin{bmatrix}\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D) & 0 \\ 0 & \text{trun}(D^*)^{\mathsf{T}}\,\text{trun}(D^*)\end{bmatrix}$$

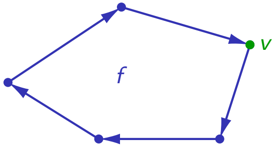
$$= \det\left(\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D)\right)\cdot\det\left(\text{trun}(D^*)^{\mathsf{T}}\,\text{trun}(D^*)\right)$$

$$= \tau(G)\cdot\tau(G^*) = \tau(G)^2.$$

Proof of Hope. Let c be a column of $\text{trun}(D)$ corresponding to vertex v of D. Let c' be a column of $\text{trun}(D^*)$ corresponding to face f of D. Want: $c^{\mathsf{T}}c' = 0$.

If there is no arc incident to both v and f, done.

Otherwise,



Proof (cont'd)

Hope

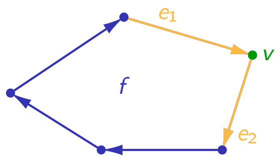
$\text{trun}(D)^\mathsf{T}\,\text{trun}(D^*) = 0$, so

$$\left(\det\left[\ \text{trun}(D)\ \middle|\ \text{trun}(D^*)\ \right]\right)^2 = \det\begin{bmatrix}\text{trun}(D)^\mathsf{T}\text{trun}(D) & 0 \\ 0 & \text{trun}(D^*)^\mathsf{T}\text{trun}(D^*)\end{bmatrix}$$
$$= \det\left(\text{trun}(D)^\mathsf{T}\text{trun}(D)\right)\cdot\det\left(\text{trun}(D^*)^\mathsf{T}\text{trun}(D^*)\right)$$
$$= \tau(G)\cdot\tau(G^*) = \tau(G)^2.$$

Proof of Hope. Let c be a column of $\text{trun}(D)$ corresponding to vertex v of D. Let c' be a column of $\text{trun}(D^*)$ corresponding to face f of D. Want: $c^\mathsf{T}c' = 0$.

If there is no arc incident to both v and f, done.

Otherwise,



$$\begin{matrix} & v \\ e_1 & \\ e_2 & \\ & \end{matrix}\begin{bmatrix}-1 \\ 1 \\ 0 \\ \vdots \\ 0\end{bmatrix}$$

Proof (cont'd)

Hope

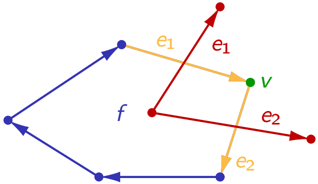
$\mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}(D^*) = 0$, so

$$\left(\det\left[\ \mathrm{trun}(D)\ \middle|\ \mathrm{trun}(D^*)\ \right]\right)^2 = \det\begin{bmatrix}\mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}(D) & 0\\ 0 & \mathrm{trun}(D^*)^{\mathsf{T}}\,\mathrm{trun}(D^*)\end{bmatrix}$$
$$= \det\left(\mathrm{trun}(D)^{\mathsf{T}}\,\mathrm{trun}(D)\right)\cdot\det\left(\mathrm{trun}(D^*)^{\mathsf{T}}\,\mathrm{trun}(D^*)\right)$$
$$= \tau(G)\cdot\tau(G^*) = \tau(G)^2.$$

Proof of Hope. Let c be a column of $\mathrm{trun}(D)$ corresponding to vertex v of D. Let c' be a column of $\mathrm{trun}(D^*)$ corresponding to face f of D. Want: $c^{\mathsf{T}}c' = 0$.

If there is no arc incident to both v and f, done.

Otherwise,



$$\begin{matrix} & v \\ e_1 & \\ e_2 & \\ & \end{matrix}\begin{bmatrix}-1\\1\\0\\\vdots\\0\end{bmatrix}\cdot\begin{matrix} & f \\ e_1 & \\ e_2 & \\ & \end{matrix}\begin{bmatrix}-1\\-1\\0\\\vdots\\0\end{bmatrix}$$

Proof (cont'd)

Hope

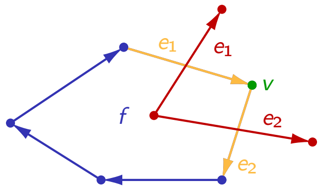
$\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D^*) = 0$, so

$$\left(\det\left[\ \text{trun}(D)\ \middle|\ \text{trun}(D^*)\ \right]\right)^2 = \det\begin{bmatrix}\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D) & 0 \\ 0 & \text{trun}(D^*)^{\mathsf{T}}\,\text{trun}(D^*)\end{bmatrix}$$
$$= \det\left(\text{trun}(D)^{\mathsf{T}}\,\text{trun}(D)\right)\cdot\det\left(\text{trun}(D^*)^{\mathsf{T}}\,\text{trun}(D^*)\right)$$
$$= \tau(G)\cdot\tau(G^*) = \tau(G)^2.$$

Proof of Hope. Let c be a column of $\text{trun}(D)$ corresponding to vertex v of D. Let c' be a column of $\text{trun}(D^*)$ corresponding to face f of D. Want: $c^{\mathsf{T}}c' = 0$.

If there is no arc incident to both v and f, done.

Otherwise,



$$\begin{matrix} & v \\ e_1 & \\ e_2 & \end{matrix}\begin{bmatrix}-1\\1\\0\\\vdots\\0\end{bmatrix}\cdot\begin{matrix} & f \\ e_1 & \\ e_2 & \end{matrix}\begin{bmatrix}-1\\-1\\0\\\vdots\\0\end{bmatrix} = 0$$

Case of nonplanar graphs

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ *and any* $e \in E$ *with* $G \setminus e$ *connected, we have* $\mathrm{maxdet}(G) \leq \mathrm{maxdet}(G/e) + \mathrm{maxdet}(G \setminus e)$.

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ *and any* $e \in E$ *with* $G \setminus e$ *connected, we have* $\text{maxdet}(G) \leq \text{maxdet}(G/e) + \text{maxdet}(G \setminus e)$.

Reminiscent?

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ *and any* $e \in E$ *with* $G \setminus e$ *connected, we have* $\text{maxdet}(G) \leq \text{maxdet}(G/e) + \text{maxdet}(G \setminus e)$.

Reminiscent?

Proposition (deletion-contraction relation for the number of spanning trees)

For any graph $G = (V, E)$ *and any* $e \in E$*, we have* $\tau(G) = \tau(G/e) + \tau(G \setminus e)$.

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ *and any* $e \in E$ *with* $G \setminus e$ *connected, we have* $\mathrm{maxdet}(G) \leq \mathrm{maxdet}(G/e) + \mathrm{maxdet}(G \setminus e)$.

Reminiscent?

Proposition (deletion-contraction relation for the number of spanning trees)

For any graph $G = (V, E)$ *and any* $e \in E$*, we have* $\tau(G) = \tau(G/e) + \tau(G \setminus e)$.

Merge-cut lemma

For any connected graph $G = (V, E)$ and any $e \in E$ with $G \setminus e$ connected, we have $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e)$.

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ *and any* $e \in E$ *with* $G \setminus e$ *connected, we have* $\mathrm{maxdet}(G) \leq \mathrm{maxdet}(G/e) + \mathrm{maxdet}(G \setminus e)$.

Reminiscent?

Proposition (deletion-contraction relation for the number of spanning trees)

For any graph $G = (V, E)$ *and any* $e \in E$*, we have* $\tau(G) = \tau(G/e) + \tau(G \setminus e)$.

Merge-cut lemma

For any connected graph $G = (V, E)$ and any $e \in E$ with $G \setminus e$ connected, we have $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e)$.

Corollary

For any graph G*, we have* $\varepsilon(G) \geq 0$.

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ *and any* $e \in E$ *with* $G \setminus e$ *connected, we have* $\mathrm{maxdet}(G) \leq \mathrm{maxdet}(G/e) + \mathrm{maxdet}(G \setminus e)$.

Reminiscent?

Proposition (deletion-contraction relation for the number of spanning trees)

For any graph $G = (V, E)$ *and any* $e \in E$*, we have* $\tau(G) = \tau(G/e) + \tau(G \setminus e)$.

Merge-cut lemma

For any connected graph $G = (V, E)$ and any $e \in E$ with $G \setminus e$ connected, we have $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e)$.

Corollary

For any graph G*, we have* $\varepsilon(G) \geq 0$.

Proof.

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ *and any* $e \in E$ *with* $G \setminus e$ *connected, we have* $\operatorname{maxdet}(G) \leq \operatorname{maxdet}(G/e) + \operatorname{maxdet}(G \setminus e)$.

Reminiscent?

Proposition (deletion-contraction relation for the number of spanning trees)

For any graph $G = (V, E)$ *and any* $e \in E$*, we have* $\tau(G) = \tau(G/e) + \tau(G \setminus e)$.

Merge-cut lemma

For any connected graph $G = (V, E)$ and any $e \in E$ with $G \setminus e$ connected, we have $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e)$.

Corollary

For any graph G*, we have* $\varepsilon(G) \geq 0$.

Proof. If G is a tree, $\varepsilon(G) = 0$.

Case of nonplanar graphs

Lemma (merge-cut lemma)

For any connected graph $G = (V, E)$ and any $e \in E$ with $G \setminus e$ connected, we have $\mathrm{maxdet}(G) \leq \mathrm{maxdet}(G/e) + \mathrm{maxdet}(G \setminus e)$.

Reminiscent?

Proposition (deletion-contraction relation for the number of spanning trees)

For any graph $G = (V, E)$ and any $e \in E$, we have $\tau(G) = \tau(G/e) + \tau(G \setminus e)$.

Merge-cut lemma

For any connected graph $G = (V, E)$ and any $e \in E$ with $G \setminus e$ connected, we have $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e)$.

Corollary

For any graph G, we have $\varepsilon(G) \geq 0$.

Proof. If G is a tree, $\varepsilon(G) = 0$. Otherwise, G has a non-bridge edge e, so we induct.

Case of nonplanar graphs (cont'd)

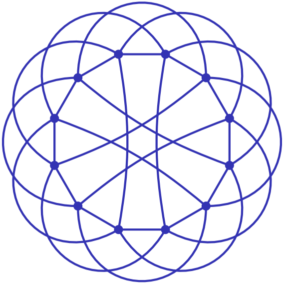
Theorem (Wagner '1937)

A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.

Case of nonplanar graphs (cont'd)

Theorem (Wagner '1937)

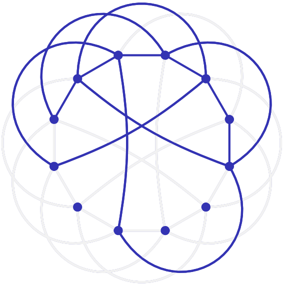
A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.



Case of nonplanar graphs (cont'd)

Theorem (Wagner '1937)

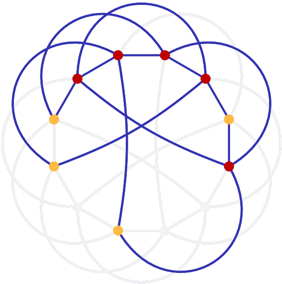
A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.



Case of nonplanar graphs (cont'd)

Theorem (Wagner '1937)

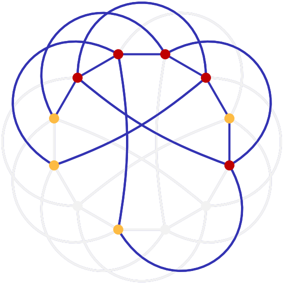
A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.

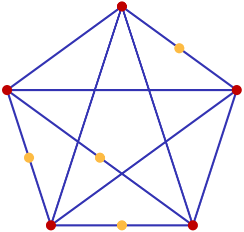


Case of nonplanar graphs (cont'd)

Theorem (Wagner '1937)

A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.

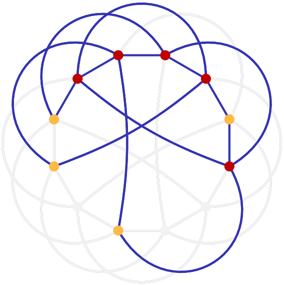


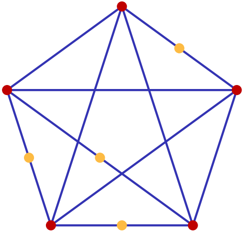


Case of nonplanar graphs (cont'd)

Theorem (Wagner '1937)

A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of *edge contractions* *and* *edge deletions*.



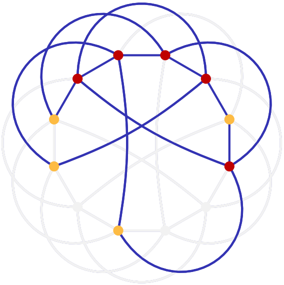


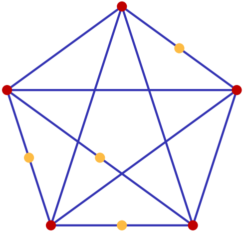
- Each intermediate graph is connected!

Case of nonplanar graphs (cont'd)

Theorem (Wagner '1937)

A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.



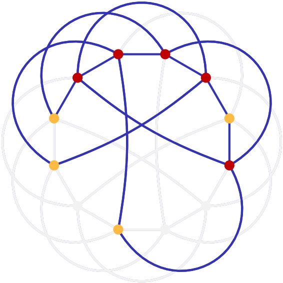


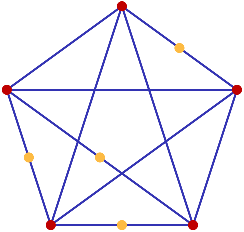
- Each intermediate graph is connected!
- If e is non-bridge, then $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e) \geq \max\{\varepsilon(G/e), \varepsilon(G \setminus e)\}$.

Case of nonplanar graphs (cont'd)

Theorem (Wagner '1937)

A graph is nonplanar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.





- Each intermediate graph is connected!
- If e is non-bridge, then $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e) \geq \max\{\varepsilon(G/e), \varepsilon(G \setminus e)\}$.
- Inductively, $\varepsilon(G) \geq \min\{\varepsilon(K_5), \varepsilon(K_{3,3})\}$ if G is nonplanar.

a question from polyhedral combinatorics

Totally unimodular matrices

Definition

A matrix is totally unimodular if any square submatrix has determinant in $\{-1, 0, 1\}$.

Totally unimodular matrices

Definition

A matrix is totally unimodular if any square submatrix has determinant in $\{-1, 0, 1\}$.

Total unimodularity is an important tool for analyzing integrality of a polyhedron.

Totally unimodular matrices

Definition

A matrix is totally unimodular if any square submatrix has determinant in $\{-1, 0, 1\}$.

Total unimodularity is an important tool for analyzing integrality of a polyhedron.

Proposition (Poincaré '1900; Veblen, Franklin '1921; Chuard '1922)

The incidence matrix of any digraph is totally unimodular.

Totally unimodular matrices

Definition

A matrix is totally unimodular if any square submatrix has determinant in $\{-1, 0, 1\}$.

Total unimodularity is an important tool for analyzing integrality of a polyhedron.

Proposition (Poincaré '1900; Veblen, Franklin '1921; Chuard '1922)

The incidence matrix of any digraph is totally unimodular.

Is the concatenation $[M|N]$ of two incidence matrices M and N also totally unimodular?

Totally unimodular matrices

Definition

A matrix is totally unimodular if any square submatrix has determinant in $\{-1, 0, 1\}$.

Total unimodularity is an important tool for analyzing integrality of a polyhedron.

Proposition (Poincaré '1900; Veblen, Franklin '1921; Chuard '1922)

The incidence matrix of any digraph is totally unimodular.

Is the concatenation $[M|N]$ of two incidence matrices M and N also totally unimodular?

No, $\det \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & -1 & 1 \end{array}\right] = -3.$

Totally unimodular matrices

Definition

A matrix is totally unimodular if any square submatrix has determinant in $\{-1, 0, 1\}$.

Total unimodularity is an important tool for analyzing integrality of a polyhedron.

Proposition (Poincaré '1900; Veblen, Franklin '1921; Chuard '1922)

The incidence matrix of any digraph is totally unimodular.

Is the concatenation $[M|N]$ of two incidence matrices M and N also totally unimodular?

No, $\det \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & -1 & 1 \end{array}\right] = -3.$

How large/small can the determinant of a square submatrix of such a concatenation?

Two extremal problems

Two extremal problems

Definition

For all $m \in \mathbb{N}$, let Δ_m be the maximum determinant of an $m \times m$ concatenation $[M|N]$ of two incidence submatrices M and N.

Two extremal problems

Definition

For all $m \in \mathbb{N}$, let Δ_m be the maximum determinant of an $m \times m$ concatenation $[M|N]$ of two incidence submatrices M and N.

Definition (Kenyon '1996)

For all $m \in \mathbb{N}$, let T_m be the maximum number of spanning trees in a planar graph with m edges, where we allow parallel edges between two vertices.

Two extremal problems

Definition

For all $m \in \mathbb{N}$, let Δ_m be the maximum determinant of an $m \times m$ concatenation $[M|N]$ of two incidence submatrices M and N.

Definition (Kenyon '1996)

For all $m \in \mathbb{N}$, let T_m be the maximum number of spanning trees in a planar graph with m edges, where we allow parallel edges between two vertices.

Theorem (Bu, **P.** '2024+)

For any planar graph G, we have $\mathrm{maxdet}(G) = \tau(G)$.

Two extremal problems

Definition

For all $m \in \mathbb{N}$, let Δ_m be the maximum determinant of an $m \times m$ concatenation $[M|N]$ of two incidence submatrices M and N.

Definition (Kenyon '1996)

For all $m \in \mathbb{N}$, let T_m be the maximum number of spanning trees in a planar graph with m edges, where we allow parallel edges between two vertices.

Theorem (Bu, **P.** '2024+)

For any planar graph G, we have $\text{maxdet}(G) = \tau(G)$.

Corollary

For all $m \in \mathbb{N}$, we have $T_m \leq \Delta_m$.

An upper bound

An upper bound

Theorem (Bu, P. '2024+)

For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

An upper bound

Theorem (Bu, P. '2024+)

For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

Proof.

An upper bound

Theorem (Bu, P. '2024+)

For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

Proof. Let $[M|N]$ achieve Δ_m.

An upper bound

Theorem (Bu, P. '2024+)

For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

Proof. Let $[M|N]$ achieve Δ_m. Extend $[M|N]$ to $[M'|N']$ so that each nonzero row of M and N has exactly one 1 and exactly one -1.

An upper bound

Theorem (Bu, P. '2024+)

For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

Proof. Let $[M|N]$ achieve Δ_m. Extend $[M|N]$ to $[M'|N']$ so that each nonzero row of M and N has exactly one 1 and exactly one -1. Then $[M'|N']$ has $m+2$ columns with at most $4m$ nonzero entries.

An upper bound

Theorem (Bu, P. '2024+)

For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

Proof. Let $[M|N]$ achieve Δ_m. Extend $[M|N]$ to $[M'|N']$ so that each nonzero row of M and N has exactly one 1 and exactly one -1. Then $[M'|N']$ has $m+2$ columns with at most $4m$ nonzero entries.

Then $[M'|N']$ has a column with at most $\lfloor 4m/(m+2) \rfloor \leq 3$ nonzero entries.

An upper bound

Theorem (Bu, P. '2024+)

For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

Proof. Let $[M|N]$ achieve Δ_m. Extend $[M|N]$ to $[M'|N']$ so that each nonzero row of M and N has exactly one 1 and exactly one -1. Then $[M'|N']$ has $m+2$ columns with at most $4m$ nonzero entries.

Then $[M'|N']$ has a column with at most $\lfloor 4m/(m+2) \rfloor \leq 3$ nonzero entries. By elementary column operations, w.l.o.g., this column is in $[M|N]$.

An upper bound

Theorem (Bu, P. '2024+)

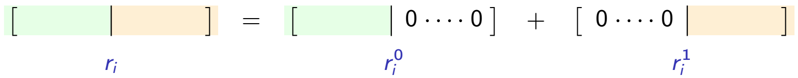
For $m \in \mathbb{N}$, $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

Proof. Let $[M|N]$ achieve Δ_m. Extend $[M|N]$ to $[M'|N']$ so that each nonzero row of M and N has exactly one 1 and exactly one -1. Then $[M'|N']$ has $m+2$ columns with at most $4m$ nonzero entries.

Then $[M'|N']$ has a column with at most $\lfloor 4m/(m+2) \rfloor \leq 3$ nonzero entries. By elementary column operations, w.l.o.g., this column is in $[M|N]$.

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} a_{11} & \\ a_{21} & \\ a_{31} & \\ 0 & \\ \vdots & \\ 0 & \end{array} \right]$$

An upper bound (cont'd)



An upper bound (cont'd)

$$\underbrace{[\quad \mid \quad]}_{r_i} = \underbrace{[\quad \mid 0 \cdots\cdots 0\,]}_{r_i^0} + \underbrace{[\,0 \cdots\cdots 0 \mid \quad]}_{r_i^1}$$

- By multilinearity of determinants,

$$\det \left[\; M \;\middle|\; N \;\right] = \sum_{\alpha,\beta,\gamma \in \{0,1\}} \det \left[r_1^{\alpha} \quad r_2^{\beta} \quad r_3^{\gamma} \quad r_4 \cdots\cdots r_m \right]^{\mathsf{T}}.$$

An upper bound (cont'd)

$$\underbrace{\left[\quad \middle| \quad \right]}_{r_i} = \underbrace{\left[\quad \middle| 0 \cdots\cdots 0 \right]}_{r_i^0} + \underbrace{\left[0 \cdots\cdots 0 \middle| \quad \right]}_{r_i^1}$$

- By multilinearity of determinants,

$$\det \left[\; M \;\middle|\; N \;\right] = \sum_{\alpha,\beta,\gamma \in \{0,1\}} \det \left[r_1^\alpha \quad r_2^\beta \quad r_3^\gamma \quad r_4 \cdots\cdots r_m \right]^\mathsf{T}.$$

- Case 1: $\alpha = \beta = \gamma = 1$.

An upper bound (cont'd)

$$\underbrace{\left[\quad\;\middle|\quad\;\right]}_{r_i} = \underbrace{\left[\quad\;\middle|\; 0 \cdots\cdots 0\right]}_{r_i^0} + \underbrace{\left[0 \cdots\cdots 0 \;\middle|\quad\;\right]}_{r_i^1}$$

- By multilinearity of determinants,

$$\det\left[\; M \;\middle|\; N \;\right] = \sum_{\alpha,\beta,\gamma\in\{0,1\}} \det\left[r_1^{\alpha} \quad r_2^{\beta} \quad r_3^{\gamma} \quad r_4 \cdots\cdots\cdots r_m\right]^{\mathsf{T}}.$$

- Case 1: $\alpha = \beta = \gamma = 1$.

$$\begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^1 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ 0 & \\ \vdots & \\ 0 & \end{array}\right]$$

An upper bound (cont'd)

$$\underbrace{\left[\quad\middle|\quad\right]}_{r_i} = \underbrace{\left[\quad\middle| 0 \cdots\cdots 0\right]}_{r_i^0} + \underbrace{\left[0 \cdots\cdots 0 \middle|\quad\right]}_{r_i^1}$$

- By multilinearity of determinants,

$$\det\left[\; M \;\middle|\; N \;\right] = \sum_{\alpha,\beta,\gamma\in\{0,1\}} \det\left[r_1^{\alpha} \quad r_2^{\beta} \quad r_3^{\gamma} \quad r_4 \cdots\cdots\cdots r_m\right]^{\mathsf{T}}.$$

- Case 1: $\alpha = \beta = \gamma = 1$.

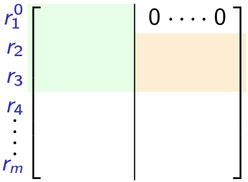
$$\det \begin{matrix} r_1^1 \\ r_2^1 \\ r_3^1 \\ r_4 \\ \vdots \\ r_m \end{matrix} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ 0 & \\ \vdots & \\ 0 & \end{array}\right] = 0.$$

An upper bound (cont'd)

- Case 2: $\alpha = 0$.

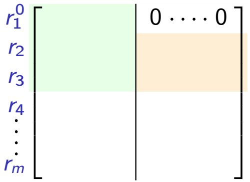
An upper bound (cont'd)

- Case 2: $\alpha = 0$.



An upper bound (cont'd)

- Case 2: $\alpha = 0$.

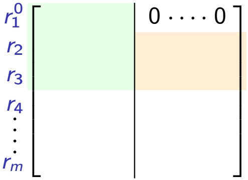


Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only,

An upper bound (cont'd)

- Case 2: $\alpha = 0$.

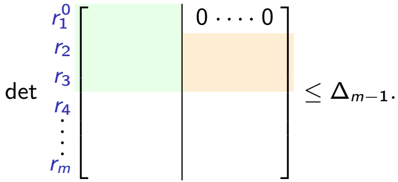


Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

An upper bound (cont'd)

- Case 2: $\alpha = 0$.



Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

An upper bound (cont'd)

Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

- Case 3: $\alpha = 1$ and $\beta = 0$.

An upper bound (cont'd)

Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

- Case 3: $\alpha = 1$ and $\beta = 0$.

$$\begin{array}{c} r_1^1 \\ r_2^0 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \end{array}\right]$$

An upper bound (cont'd)

Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

- Case 3: $\alpha = 1$ and $\beta = 0$.

$$\det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots 0 & \\ & 0 \cdots 0 \\ & \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-2}.$$

An upper bound (cont'd)

Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

- Case 4: $\alpha = \beta = 1$ and $\gamma = 0$.

An upper bound (cont'd)

Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

- Case 4: $\alpha = \beta = 1$ and $\gamma = 0$.

$$\begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^0 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \end{array}\right]$$

An upper bound (cont'd)

Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

- Case 4: $\alpha = \beta = 1$ and $\gamma = 0$.

$$\det \begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^0 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-3}.$$

An upper bound (cont'd)

Lemma (Bu, P. '2024+; informal)

If a square bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain a square bi-incidence matrix with one fewer row and column, while preserving the determinant.

- Case 4: $\alpha = \beta = 1$ and $\gamma = 0$.

$$\det \begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^0 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-3}.$$

- $\Delta_m \leq \Delta_{m-1} + \Delta_{m-2} + \Delta_{m-3}$.

a “wild” conjecture

A "wild" conjecture

Evidence 1

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

A "wild" conjecture

Evidence 1

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

Evidence 2 (Stoimenow '2007)

For $m \in \mathbb{N}$, $T_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

A "wild" conjecture

Evidence 1

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

Evidence 2 (Stoimenow '2007)

For $m \in \mathbb{N}$, $T_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

The argument of Stoimenow (2007) is knot-theoretic and (in an equivalent way) uses the fact that any triangle-free planar graph has a vertex of degree at most 3.

A "wild" conjecture

Evidence 1

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

Evidence 2 (Stoimenow '2007)

For $m \in \mathbb{N}$, $T_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

The argument of Stoimenow (2007) is knot-theoretic and (in an equivalent way) uses the fact that any triangle-free planar graph has a vertex of degree at most 3.

Conjecture 1

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

A "wild" conjecture

Evidence 1

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

Evidence 2 (Stoimenow '2007)

For $m \in \mathbb{N}$, $T_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

The argument of Stoimenow (2007) is knot-theoretic and (in an equivalent way) uses the fact that any triangle-free planar graph has a vertex of degree at most 3.

Conjecture 1

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

Fact (Wu '1977; Kenyon '1996)

For all $m \in \mathbb{N}$, we have $T_m \geq 1.7916^m$, which is achieved by square grid graphs.

Towards the conjecture

Conjecture 1

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

Towards the conjecture

Conjecture 1

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

Conjecture 2

If G is a connected nonplanar graph with m edges, then $\text{maxdet}(G) \leq T_m$.

Towards the conjecture

Conjecture 1

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

Conjecture 2

If G is a connected nonplanar graph with m edges, then $\text{maxdet}(G) \leq T_m$.

- In other words: the maximum $\text{maxdet}(\cdot)$ value over graphs with the same number of edges is always achieved by planar graphs.

Towards the conjecture

Conjecture 1

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

Conjecture 2

If G is a connected nonplanar graph with m edges, then $\mathrm{maxdet}(G) \leq T_m$.

- In other words: the maximum $\mathrm{maxdet}(\cdot)$ value over graphs with the same number of edges is always achieved by planar graphs.
- In other words: nonplanar graphs are "dominated" by the best planar graph with the same number of edges linear-algebraically in terms of their $\mathrm{maxdet}(\cdot)$ values.

Towards the conjecture

Conjecture 1

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

Conjecture 2

If G is a connected nonplanar graph with m edges, then $\text{maxdet}(G) \leq T_m$.

- In other words: the maximum maxdet$(\cdot)$ value over graphs with the same number of edges is always achieved by planar graphs.
- In other words: nonplanar graphs are "dominated" by the best planar graph with the same number of edges linear-algebraically in terms of their maxdet$(\cdot)$ values.
- Conjectures 1 and 2 are equivalent by the zero-row-removal lemma!

Towards the conjecture (cont'd)

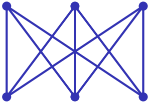
Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\mathrm{maxdet}(G) \leq T_m$.

Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\mathrm{maxdet}(G) \leq T_m$.

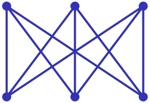


$\mathrm{maxdet} = 63$

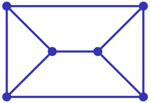
Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\text{maxdet}(G) \le T_m$.



$\text{maxdet} = 63$

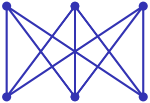


$\text{maxdet} = 75$

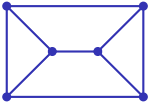
Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

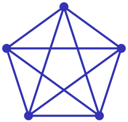
If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\mathrm{maxdet}(G) \leq T_m$.



$\mathrm{maxdet} = 63$



$\mathrm{maxdet} = 75$

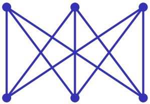


$\mathrm{maxdet} = 100$

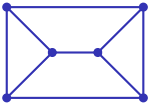
Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

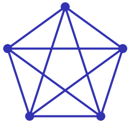
If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\mathrm{maxdet}(G) \leq T_m$.



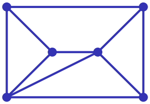
$\mathrm{maxdet} = 63$



$\mathrm{maxdet} = 75$



$\mathrm{maxdet} = 100$



$\mathrm{maxdet} = 130$

Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\text{maxdet}(G) \leq T_m$.

Proof ideas.

Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\mathrm{maxdet}(G) \leq T_m$.

Proof ideas.

- Consider the bi-incidence matrix that attains $\mathrm{maxdet}(G)$. It suffices to show that either the left side or the right side is "planar."

Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\text{maxdet}(G) \leq T_m$.

Proof ideas.

- Consider the bi-incidence matrix that attains maxdet(G). It suffices to show that either the left side or the right side is "planar."
- In the case of $K_{3,3}$, we show that the right side is "planar" w.l.o.g., using convexity.

Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\text{maxdet}(G) \le T_m$.

Proof ideas.

- Consider the bi-incidence matrix that attains $\text{maxdet}(G)$. It suffices to show that either the left side or the right side is "planar."
- In the case of $K_{3,3}$, we show that the right side is "planar" w.l.o.g., using convexity.
- In the case of K_5, we show that the left side (a truncated incidence matrix of G) can be transformed to be "planar" using careful matrix operations, while preserving the determinant.

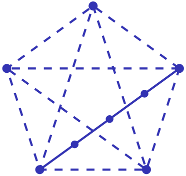
Towards the conjecture (cont'd)

Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\mathrm{maxdet}(G) \leq T_m$.

Proof ideas.

- Consider the bi-incidence matrix that attains $\mathrm{maxdet}(G)$. It suffices to show that either the left side or the right side is "planar."
- In the case of $K_{3,3}$, we show that the right side is "planar" w.l.o.g., using convexity.
- In the case of K_5, we show that the left side (a truncated incidence matrix of G) can be transformed to be "planar" using careful matrix operations, while preserving the determinant.



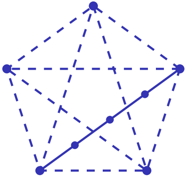
Towards the conjecture (cont'd)

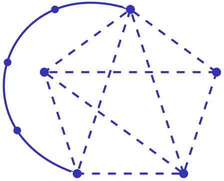
Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\text{maxdet}(G) \leq T_m$.

Proof ideas.

- Consider the bi-incidence matrix that attains $\text{maxdet}(G)$. It suffices to show that either the left side or the right side is "planar."
- In the case of $K_{3,3}$, we show that the right side is "planar" w.l.o.g., using convexity.
- In the case of K_5, we show that the left side (a truncated incidence matrix of G) can be transformed to be "planar" using careful matrix operations, while preserving the determinant.





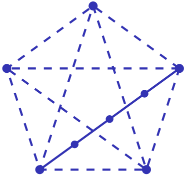
Towards the conjecture (cont'd)

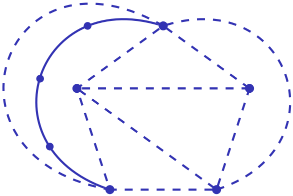
Theorem (Bu, P. '2024+)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\mathrm{maxdet}(G) \leq T_m$.

Proof ideas.

- Consider the bi-incidence matrix that attains $\mathrm{maxdet}(G)$. It suffices to show that either the left side or the right side is "planar."
- In the case of $K_{3,3}$, we show that the right side is "planar" w.l.o.g., using convexity.
- In the case of K_5, we show that the left side (a truncated incidence matrix of G) can be transformed to be "planar" using careful matrix operations, while preserving the determinant.





Future directions

Future directions

- Does Conjecture 1 hold?

Future directions

- Does Conjecture 1 hold?
  - If so, what are the asymptotic behaviors of T_m and Δ_m?

Future directions

- Does Conjecture 1 hold?
  - If so, what are the asymptotic behaviors of T_m and Δ_m?
- Can the ideas from the last proof, e.g., the edge relocation method, be generalized to a broader class of nonplanar graphs?

Future directions

- Does Conjecture 1 hold?
  - If so, what are the asymptotic behaviors of T_m and Δ_m?
- Can the ideas from the last proof, e.g., the edge relocation method, be generalized to a broader class of nonplanar graphs?
  - How can one deal with many copies of subdivisions of K_5 and $K_{3,3}$?

Future directions

- Does Conjecture 1 hold?
  - If so, what are the asymptotic behaviors of T_m and Δ_m?
- Can the ideas from the last proof, e.g., the edge relocation method, be generalized to a broader class of nonplanar graphs?
  - How can one deal with many copies of subdivisions of K_5 and $K_{3,3}$?
- Faster algorithms for counting spanning trees in a planar graph?

Future directions

- Does Conjecture 1 hold?
  - If so, what are the asymptotic behaviors of T_m and Δ_m?
- Can the ideas from the last proof, e.g., the edge relocation method, be generalized to a broader class of nonplanar graphs?
  - How can one deal with many copies of subdivisions of K_5 and $K_{3,3}$?
- Faster algorithms for counting spanning trees in a planar graph?
- Faster algorithms for testing planarity?

Thanks!

It's nice to be back.