The Tales of Planar Graphs

The Extremal Problems, the Criterion, and the Conjectures

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Meet-and-Think
CMU Tepper School of Business

Disclaimer: I am **not** a combinatorialist.

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some `StackExchange` answer (user `JimT`):

> This is a very, very tough question. It is obvious that for any graph G with n edges the number of spanning trees $t(G)$ does not exceed 2^n (each edge is either included in or excluded from a subtree; this is also the upper bound of the number of connected subgraphs of G). We can somewhat improve this - if G has m vertices ($m \leqslant n+1$) then we have
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> $$t(G) \leqslant \binom{n}{m-1} < 2^n,$$
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> since we need to choose $m-1$ edges out of n (not arbitrarily, of course).

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a **lower bound**: $T_m \geq 1.7916^m$ achieved by square grid graphs some **upper bounds** ($T_m \leq \tau^m$):

2 As a matter of fact, the conjecture above has a relatively elementary proof (found by yours truly) for $\tau \approx 1.96$. Of course, that is not very close to the hypothetical upper bound of $e^{2C/\pi}$, but at least it is something... – JimT Jan 8, 2021 at 4:21

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Actually, it is now at 1.8637^n. See arxiv.org/abs/2103.10523 – JimT Aug 3, 2024 at 3:34

Quick update: Here is a result by Stoimenow in knot theory, which, in their abstract, mentions giving an upper bound of 1.8393n on this number. Here: projecteuclid.org/journals/tokyo-journal-of-mathematics/... – Alien Sep 28, 2024 at 0:17

@Alien Nice! One comment though - I quickly looked thru the article; it is very non-elementary, as it uses a bunch of other results in links and knot theory, as well as related polynomial algebra. But still, it proves the even lower upper bound, so the gap between this and 1.7916... is getting smaller with every post here :) . – JimT Sep 29, 2024 at 2:32

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theorem (Stoimenow 2007)

$T_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

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... and also some nice by-products on planarity criteria from our techniques.

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$$\Delta_m \leq \delta^m$$

definitions

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$\implies$ $[M'|N']$ has a column with at most $\lfloor 4m/(m+2) \rfloor = 3$ nonzero entries.

W.l.o.g. (by column operations), this column is in $[M|N]$.

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} a_{11} & \\ a_{21} & \\ a_{31} & \\ 0 & \\ \vdots & \\ 0 & \end{array} \right]$$

proof of $\Delta_m \leq \delta^m$ (cont'd)

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\begin{array}{c}
r_1 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m
\end{array}
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\vdots & \\
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$$

$$
\underset{r_i}{\left[\begin{array}{c|c} \quad\quad\quad & \quad\quad\quad \end{array}\right]}
=
\underset{r_i^0}{\left[\begin{array}{c|c} \quad\quad\quad & 0 \cdots\cdots 0 \end{array}\right]}
+
\underset{r_i^1}{\left[\begin{array}{c|c} 0 \cdots\cdots 0 & \quad\quad\quad \end{array}\right]}
$$

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\vdots & \\
0 &
\end{array}\right]
$$

$$
\underset{r_i}{\left[\begin{array}{c|c} \quad & \quad \end{array}\right]} = \underset{r_i^0}{\left[\begin{array}{c|c} \quad & 0 \cdots\cdots 0 \end{array}\right]} + \underset{r_i^1}{\left[\begin{array}{c|c} 0 \cdots\cdots 0 & \quad \end{array}\right]}
$$

$$
\det \left[\begin{array}{c|c} M & N \end{array}\right] = \sum_{\alpha,\beta,\gamma \in \{0,1\}} \det
\begin{array}{c}
r_1^\alpha \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m
\end{array}
\left[\begin{array}{c|c}
 & \\
 & \\
 & \\
0 & \\
\vdots & \\
0 &
\end{array}\right]
$$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^1 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ 0 \cdots\cdots 0 & \\ 0 & \\ \vdots & \\ 0 & \end{array}\right]$$

case 1: $\alpha = \beta = \gamma = 1$

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$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{array}{c} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right]$$

case 2: $\alpha = 0$

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removing-zeros lemma (Bu, **P.** 2025; informal)

If an $m \times m$ bi-incidence matrix has a row whose restriction to the left/right side has zeros only,

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If an $m \times m$ bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain an $(m-1) \times (m-1)$ square bi-incidence matrix, while preserving the determinant.

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{array}{c} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0\cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0\cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-1}.$$

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case 3: $\alpha = 1$ and $\beta = 0$

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$$\sum_{\gamma \in \{0,1\}} \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \end{array} \right] = \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \end{array} \right] \leq \Delta_{m-2}$$

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case 4: $\alpha = \beta = 1$ and $\gamma = 0$

$$\Delta_m \leq \Delta_{m-1} + \Delta_{m-2} + \Delta_{m-3}$$

$$T_m \leq \Delta_m$$

proof of $T_m \leq \Delta_m$

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices;
T_m is the maximum number of spanning trees over m-edge planar graphs.

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Goal: Given an m-edge planar graph G (achieving T_m), construct an $m \times m$ bi-incidence matrix with determinant at least the number of spanning trees in G.

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key lemma (Bu, **P.** 2025)

Let G be a connected planar graph with orientation D. Let $D^\star$ be the dual of D.

Let M and $M^\star$ be truncated incidence matrices of D and $D^\star$, of which the i^{th} rows correspond to the same arc and its dual arc for each i.

Then $|\det[M|M^\star]|$ is equal to the number of spanning trees $\tau(G)$ in G.

proof of $T_m \leq \Delta_m$ (cont'd)

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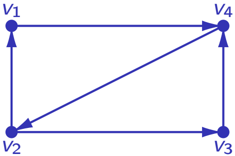

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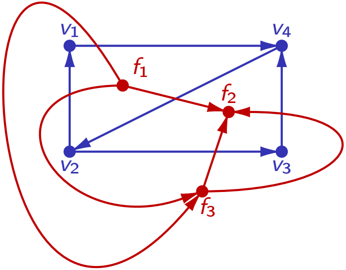

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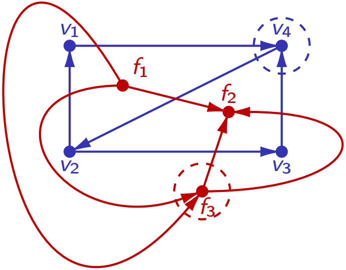

$$\begin{array}{c}
\begin{array}{ccccc} v_1 & v_2 & v_3 & f_1 & f_2 \end{array} \\
\begin{bmatrix}
-1 & & & -1 & \\
1 & -1 & & -1 & \\
 & 1 & & -1 & 1 \\
 & & -1 & & 1 \\
 & -1 & 1 & & 1
\end{bmatrix}
\end{array}$$

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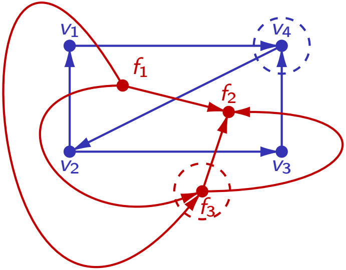

$$\begin{array}{c} \begin{matrix} v_1 & \; v_2 & \; v_3 & \; f_1 & \; f_2 \end{matrix} \\ \begin{bmatrix} -1 & & & -1 & \\ 1 & -1 & & -1 & \\ & 1 & & -1 & 1 \\ & & -1 & & 1 \\ & -1 & 1 & & 1 \end{bmatrix} \end{array}$$

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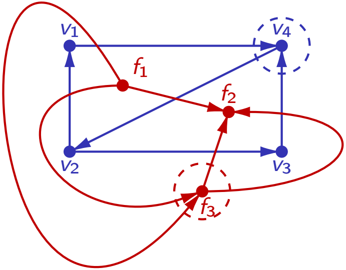

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$$

corollary

For all $m \in \mathbb{N}$, we have $T_m \leq \Delta_m$.

proof of key lemma

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$$\left(\det\left[\begin{array}{c|c} M & M^\star \end{array}\right]\right)^2 = \det\left[\begin{array}{c|c} M & M^\star \end{array}\right]^{\mathsf{T}}\left[\begin{array}{c|c} M & M^\star \end{array}\right]$$

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$$
\begin{aligned}
\left(\det\left[\; M \;\middle|\; M^\star \;\right]\right)^2 &= \det\left[\; M \;\middle|\; M^\star \;\right]^{\mathsf{T}}\left[\; M \;\middle|\; M^\star \;\right] \\
&= \det\begin{bmatrix} M^{\mathsf{T}}M & M^{\mathsf{T}}M^\star \\ (M^\star)^{\mathsf{T}}M & (M^\star)^{\mathsf{T}}M^\star \end{bmatrix}
\end{aligned}
$$

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Let G be a connected planar graph with orientation D. Let $D^\star$ be the dual of D.

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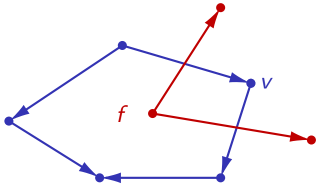

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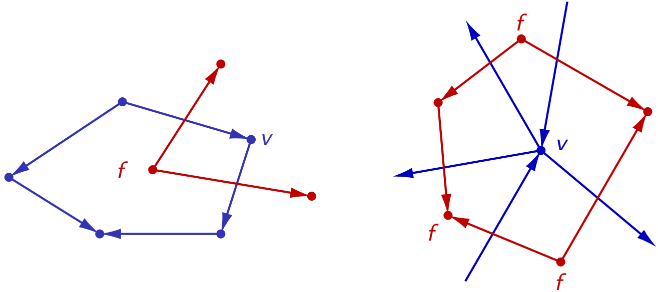

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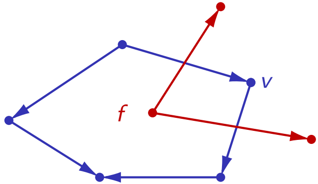

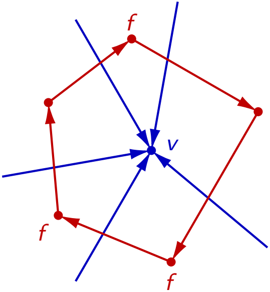

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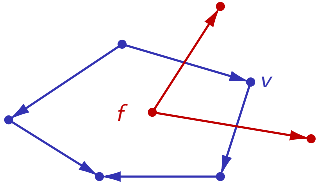

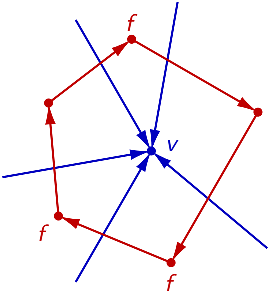

Hence, column v in M and column f in $M^{\star}$ are orthogonal.

a planarity criterion

yet another extremal problem

Given a connected graph G with m edges, how large is the maximum determinant $\Delta(G)$ over $m \times m$ matrices $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size?

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A graph is non-planar if and only if it can produce either K_5 or $K_{3,3}$ by a sequence of edge contractions and edge deletions.

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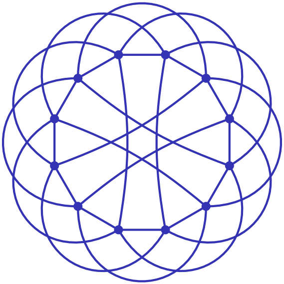

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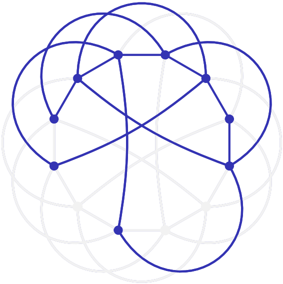

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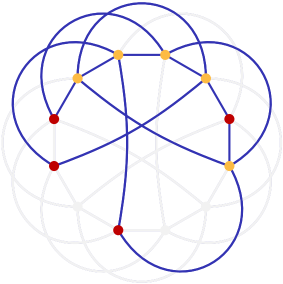

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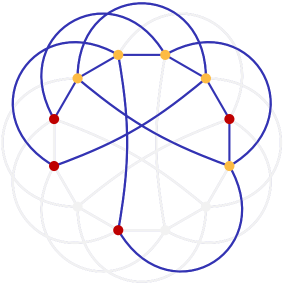

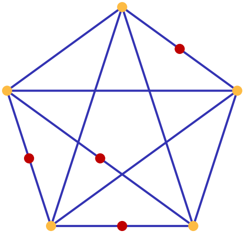

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Hence, $\varepsilon(G) \geq \min\{\varepsilon(K_{3,3}), \varepsilon(K_5)\}$.

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Open question: Are there simpler proofs?

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These bounds give us increasingly restrictive information about both the graph and properties of the bijection, which we then use to prove the desired results.

a conjecture

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T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

towards the conjecture

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The right side of P has 5 columns and at most m rows that are not all-zero, among which there are at most 9 distinct rows.

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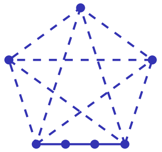

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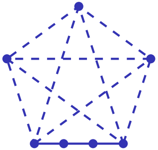

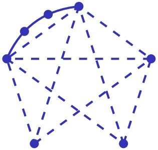

graphs of bounded genus

(ideas due to **Jack Spalding-Jamieson**)

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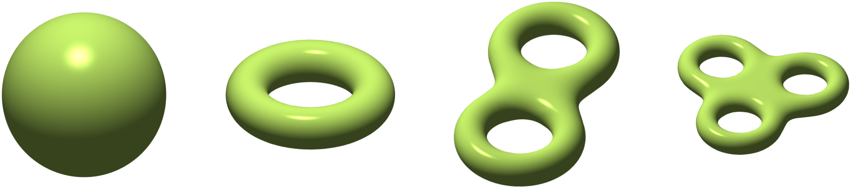

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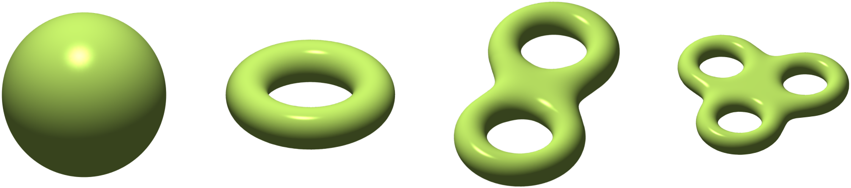

definition

The genus of a graph G is the smallest genus of a surface in which G can be embedded without crossings.

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future

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Consequences on mathematical programming?

Thanks (for listening and for the ice cream)!