# A COUNTEREXAMPLE TO BOX-HALF-INTEGRALITY OF THE INTERSECTION OF CROSSING SUBMODULAR FLOW SYSTEMS

MICHEL X. GOEMANS    YUCHONG PAN

We say that a family $\mathcal{C}$ of subsets of a ground set $V$ is *crossing* if for all $U, W \in \mathcal{C}$ with $U \cap W \neq \emptyset$ and $U \cup W \neq V$, we have $U \cap W, U \cup W \in \mathcal{C}$, and that $\mathcal{C}$ is *cross-free* if for all $U, W \in \mathcal{C}$, we have $U \subseteq W$ or $W \subseteq U$ or $U \cap W = \emptyset$ or $U \cup W = V$. We say that a set function $f : \mathcal{C} \to \mathbb{R}$ is *crossing submodular* if for all $U, W \in \mathcal{C}$ with $U \cap W \neq \emptyset$ and $U \cup W \neq V$,

$$f(U \cap W) + f(U \cup W) \leq f(U) + f(W).$$

In [1], Abdi, Cornuéjols and Zambelli proved the following theorem:

**Theorem 1** (Abdi, Cornuéjols and Zambelli, 2023, [1])**.** *Let $D = (V, A)$ be a weakly connected digraph. Let $\mathcal{C}_i$ be a crossing family over $V$ and $f_i : \mathcal{C}_i \to \mathbb{Z}$ a crossing submodular function for $i = 1, 2$. Then*

$$\begin{aligned} y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) &\leq f_1(U), && \forall U \in \mathcal{C}_1, \\ y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) &\leq f_2(U), && \forall U \in \mathcal{C}_2, \end{aligned}$$

*is totally dual integral.*

In addition, they gave an instance for which this system is not box-integral. In a talk given in the "Combinatorics and Optimization" workshop at ICERM in 2023, Abdi proposed the following conjecture:[1]

**Conjecture 2.** *Let $D = (V, A)$ be a weakly connected digraph. Let $\mathcal{C}_i$ be a crossing family over $V$ and $f_i : \mathcal{C}_i \to \mathbb{Z}$ a crossing submodular function for $i = 1, 2$. Then*

$$\begin{aligned} y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) &\leq f_1(U), && \forall U \in \mathcal{C}_1, \\ y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) &\leq f_2(U), && \forall U \in \mathcal{C}_2, \end{aligned}$$

*is box-half-integral.*

In this short note, we disprove this conjecture, making Theorem 1 more interesting.

**Theorem 3.** *There exist a weakly connected digraph $D = (V, A)$, crossing submodular functions $f_i : \mathcal{C}_i \to \mathbb{Z}$ for $i = 1, 2$ and $\ell, u \in \mathbb{Z}^A$ such that*

$$y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) \leq f_1(U), \qquad \forall U \in \mathcal{C}_1, \tag{1a}$$

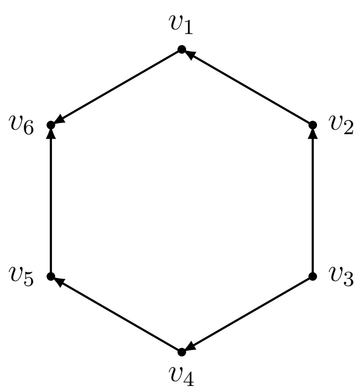
[1] See the last slide of their talk, available at `https://app.icerm.brown.edu/assets/403/4951/4951_3700_Abdi_032820231400_Slides.pdf`.

$$y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) \le f_2(U), \qquad \forall U \in \mathcal{C}_2, \tag{1b}$$

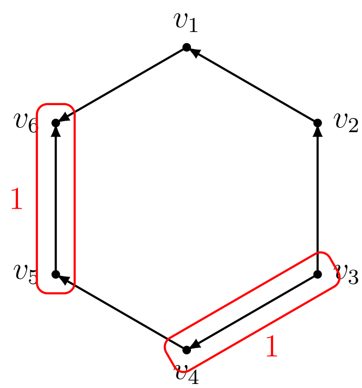
$$\ell_e \le y_e \le u_e, \qquad \forall e \in A, \tag{1c}$$

*is not half-integral.*

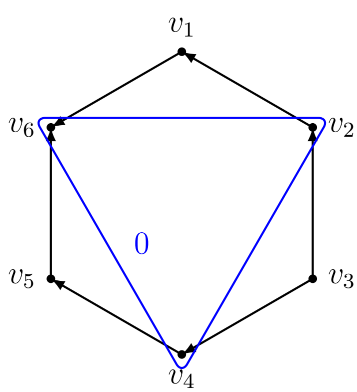
*Proof.* Let $D = (V, A)$ be the weakly connected digraph in Figure 1a. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be families of subsets of $V$ depicted in Figures 1b and 1c, with corresponding set functions $f_i : \mathcal{C}_i \to \mathbb{Z}$ for $i = 1, 2$. Since each of $\mathcal{C}_1$ and $\mathcal{C}_2$ is a family of disjoint sets, $\mathcal{C}_1$ and $\mathcal{C}_2$ are trivially crossing families, and $f_1$ and $f_2$ are trivially crossing submodular. Let $\ell = -2 \cdot \mathbf{1} \in \mathbb{Z}^A$ and $u = \mathbf{1} \in \mathbb{Z}^A$. Let $y^* \in \mathbb{Z}^A$ be the vector defined in Figure 1d. It is easy to check that $y^*$ is a feasible solution to the system (1).



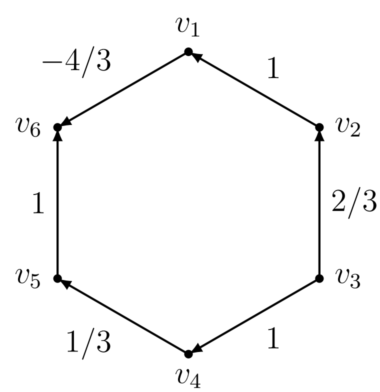

(A) The digraph $D = (V, A)$ in the proof of Theorem 3.




(B) The crossing family $\mathcal{C}_1$ and the crossing submodular set function $f_1 : \mathcal{C}_1 \to \mathbb{Z}$ in the proof of Theorem 3.




(C) The crossing family $\mathcal{C}_2$ and the crossing submodular set function $f_2 : \mathcal{C}_2 \to \mathbb{Z}$ in the proof of Theorem 3.




(D) A solution $y^* \in \mathbb{Z}^A$ to the system of linear inequalities in the proof of Theorem 3.

To see that $y^*$ is an extreme point of the polyhedron determined by the system (1), it suffices to exhibit 6 linearly independent inequalities that are tight at $y^*$. Since $y_e = u_e = 1$ for $e \in \{(v_1, v_2), (v_3, v_4), (v_5, v_6)\}$, we obtain 3 inequalities that are tight at $y^*$. In addition, it is easy to check $y(\delta^+(U)) - y(\delta^-(U)) = f_1(U) = 1$ for all $U \in \mathcal{C}_1$ and $y(\delta^+(U)) - y(\delta^-(U)) = f_2(U) = 0$ for all $U \in \mathcal{C}_2$, yielding 3 more inequalities that are tight at $y^*$. Moreover, it can be checked that these 6 inequalities are linearly independent (e.g., by computing the determinant of the matrix whose columns are these vectors). This gives 6 linearly independent inequalities that are tight at $y^*$, completing the proof. □

*Remark.* The intuition of the proof is the following. Let $M_1$ and $M_2$ be matrices with the same number of rows such that each row consists of at most one 1-entry and at most one $-1$-entry, with all other entries equal to 0. Then $M_1$ and $M_2$ are submatrices of incidence matrices of digraphs and are hence totally unimodular. Let

$$M = \left[\begin{array}{c|c} M_1 & M_2 \end{array}\right].$$

Note that $M$ is not necessarily totally unimodular, and there exist instances such that $|\det(M)|$ can be made arbitrarily large. Take such a square matrix $M$ with $|\det(M)| \geq 3$. Let $b$ be an integral vector such that $(M^{\mathrm{T}})^{-1}b$ is not half-integral. In the counterexample given in the proof,

$$M = \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 1 & -1 & 1 \end{array}\right], \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

where $\det(M) = -3$.

Let $k_1$ and $k_2$ be the numbers of columns of $M$ from $M_1$ and from $M_2$, respectively. Let $m$ be the number of rows of $M$. Let $D_0 = (V, A_0)$ be a digraph, where $V = \{0, \ldots, k_1\} \times \{0, \ldots, k_2\}$ and $A_0$ has $m$ arcs, one corresponding to each row of $M$. For each row of $M$, let $\alpha, \alpha'$ be the indices of the 1-entries from $M_1$ and from $M_2$ (or 0 if not present), respectively, and $\beta, \beta'$ the indices of $-1$-entries from $M_1$ and from $M_2$ (or 0 if not present), respectively. Then this row corresponds to the arc $((\alpha, \alpha'), (\beta, \beta'))$.

Let $S_i = \{i\} \times \{0, \ldots, k_2\}$ for each $i \in [k_1]$. Let $T_i = \{0, \ldots, k_1\} \times \{i\}$ for each $i \in [k_2]$. Let $\mathcal{C}_1 = \{S_1, \ldots, S_{k_1}\}$ and $\mathcal{C}_2 = \{T_1, \ldots, T_{k_2}\}$. Let $g_1 : \mathcal{C}_1 \to \mathbb{Z}$ be defined by $g_1(S_i) = b_i$ for all $i \in [k_1]$, and let $g_2 : \mathcal{C}_2 \to \mathbb{Z}$ be defined by $g_2(T_i) = b_{i+k_1}$ for all $i \in [k_2]$. For $i = 1, 2$, since the sets in $\mathcal{C}_i$ are disjoint, $\mathcal{C}_i$ is trivally a crossing family and $g_i$ is trivially crossing submodular. The system $M^{\mathrm{T}} y \leq b$ is exactly

$$\begin{aligned} y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) &\leq g_1(U), && \forall U \in \mathcal{C}_1, \\ y\left(\delta^+(U)\right) - y\left(\delta^-(U)\right) &\leq g_2(U), && \forall U \in \mathcal{C}_2. \end{aligned}$$

Let $y_0 \in \mathbb{Q}^A$ be the unique solution to the system $M^{\mathrm{T}} y = b$.

Now, we arbitrarily add edges to $D_0$ to create a weakly connected digraph $D = (V, A)$. For each new arc $e \in A \setminus A_0$, add a constraint $y_e \leq 1$. This results in a system $(M')^{\mathrm{T}} y \leq b'$, where

$$M' = \begin{bmatrix} M & M'' \\ 0 & I \end{bmatrix}, \qquad b' = \begin{bmatrix} b + b'' \\ \mathbf{1} \end{bmatrix},$$

where each row of $M''$ is the vector $\gamma(U) := \delta^+_{A \setminus A_0}(U) - \delta^-_{A \setminus A_0}(U)$ for $U \in \mathcal{C}_1$ and for $U \in \mathcal{C}_2$, and where each component of $b''$ is $\mathbf{1}^{\mathrm{T}} \gamma(U)$ for $U \in \mathcal{C}_1$ and for $U \in \mathcal{C}_2$. By Schur's formula, $\det(M') = \det(M) \cdot \det(I) = \det(M)$. Hence, $|\det(M')| \geq 3$. Let

$$y^* = \begin{bmatrix} y_0 \\ \mathbf{1} \end{bmatrix}.$$

It is easy to check that $y^*$ is the unique solution to the system $(M')^{\mathrm{T}} y = b$.

With a computer program and the procedure described above, one can find counterexamples to Conjecture 2 with arbitrarily large sizes.

## References

[1] Ahmad Abdi, Gérard Cornuéjols, and Giacomo Zambelli. Arc connectivity and submodular flows in digraphs. `https://www.ahmadabdi.com/papers/connectedflip.pdf`.