

Perturbation-Stable Maximum Cut

Yuchong Pan

UBC Beyond Worst-Case Analysis Reading Group
(Based on Tim Roughgarden's Notes for Stanford CS264)

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MAXIMUM CUT

Problem (MAXIMUM CUT)

Input: An undirected graph $G = (V, E)$ with edge weights $w_e > 0$ for each $e \in E$.

Goal: A cut (A, B) that maximizes the weight of the **crossing** edges.

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- ▶ MAXIMUM CUT is a type of 2-clustering problem (e.g. weights measure dissimilarities).

MAXIMUM CUT Is *NP*-Hard

Problem (MAXIMUM CUT, Decision Version)

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Output: Yes iff. there is a set $S \subseteq V$ such that the weight of the **crossing** edges is at least W .

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Input: $(c_1, \dots, c_n) \in \mathbb{Z}^n$.

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Proof Sketch (PARTITION \leq_P MAXIMUM CUT)

- ▶ $G = K_n$.
- ▶ $w_{ij} = c_i c_j$ for all $i, j \in V, i \neq j$.
- ▶ $W = \lceil \frac{1}{4} \sum c_i^2 \rceil$.

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- ▶ MINIMUM CUT is **not** *NP*-hard and can be solved by the Maximum-Flow Minimum-Cut Theorem.
- ▶ **Question:** Can't we negate the edge weights, yielding a MINIMUM CUT instance?
- ▶ No! Polynomial-time algorithms solving MINIMUM CUT require nonnegative edge weights.

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Definition (γ -Perturbation-Stability)

For $\gamma \geq 1$, an instance of MAXIMUM CUT is γ -**perturbation-stable** if a cut (A, B) is the **unique** optimal solution to all γ -**perturbations**, where each original edge weight w_e is replaced with an edge weight $w'_e \in [\frac{1}{\gamma} w_e, w_e]$.

LP Relaxation, Take 1

- ▶ **Question:** Can we use an LP relaxation similar to the one for MINIMUM CUT, i.e.

$$\begin{array}{ll} \max & \sum_{e \in E} w_e x_e \\ \text{s.t.} & x_e \geq |d_u - d_v|, \quad \forall e = uv \in E, \\ & x_e \in [0, 1], \quad \forall e \in E, \\ & d_v \in [0, 1], \quad \forall v \in V. \end{array}$$

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- ▶ **Question:** What about $x_e \leq d_u - d_v$ and $x_e \leq d_v - d_u$?
- ▶ This forces $x_e = 0$, instead of $x_e \leq |d_u - d_v|$.

LP Relaxation, Take 2

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LP Relaxation, Take 2

- ▶ Hence we obtain the LP relaxation (LP-MAXCUT):

$$\begin{array}{ll} \max & \sum_{(i,j) \in E} w_{ij} x_{ij} \\ \text{s.t.} & x_{ij} + x_{ik} + x_{jk} \leq 2, \quad \forall i, j, k \in V \text{ distinct,} \\ & x_{jk} \leq x_{ij} + x_{ik}, \quad \forall i, j, k \in V \text{ distinct,} \\ & x_{ij} \in [0, 1], \quad \forall i, j \in V \text{ distinct.} \end{array}$$

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- ▶ MINIMUM s - t CUT: $A = \{v \in V : \hat{d}_v \leq r\}$ and $B = V \setminus A$, where $r \sim \text{Uniform}(0, 1)$.
- ▶ MINIMUM MULTIWAY CUT: For each iteration, a group and a threshold are chosen uniformly randomly.

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- ▶ Since $\Delta(C) \geq 0$ and since the equality holds iff. C is an optimal cut, it follows that the randomized rounding algorithm outputs an optimal cut w.p.1.

Randomized Rounding Algorithm

Lemma

Fix an instance of the MAXIMUM CUT problem, with F^* the edges in the optimal cut, and \hat{x} the optimal solution to (LP-MAXCUT). Then there exists a randomized algorithm that generates a random cut (A, B) and a scaling parameter $\sigma > 0$ such that:

1. For every edge $e = ij \notin F^*$,

$$\mathbb{P}[e \text{ cut by } (A, B)] \geq \sigma \cdot \frac{\hat{x}_{ij}}{\alpha},$$

where $\alpha = \Theta(\log n)$;

2. For every edge $e = ij \in F^*$,

$$\mathbb{P}[e \text{ not cut by } (A, B)] \leq \sigma \cdot (1 - \hat{x}_{ij});$$

3. The rounding algorithm is deterministic iff. \hat{x} is integral.

Randomized Rounding Algorithm, Roadmap

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Proposition

Fix an instance of MAXIMUM CUT, a cut C , and a feasible solution \hat{x} to (LP-MAXCUT). For distinct $i, j \in V$, define

$$\hat{y}_{ij} = \begin{cases} \hat{x}_{ij}, & \text{if } i, j \text{ are on the same side of } C, \\ 1 - \hat{x}_{ij}, & \text{if } i, j \text{ are on different sides of } C. \end{cases}$$

Then \hat{y} satisfies the triangle inequality:

$$\hat{y}_{jk} \leq \hat{y}_{ij} + \hat{y}_{ik}$$

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Then \hat{y} satisfies the triangle inequality:

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- ▶ That is, \hat{x}, \hat{y} are both **pseudometrics** (i.e. metrics except that distinct points may have zero distances).

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Theorem (Bourgain's Theorem)

For every n -point **pseudometric** space (X, d) , there exists a randomized algorithm that generates a random partition (A, B) of X and a scaling parameter $\sigma > 0$ such that, for all distinct $i, j \in X$,

$$\mathbb{P}[i, j \text{ on different sides of } (A, B)] \in \sigma \cdot \left[\frac{d(i, j)}{\alpha}, d(i, j) \right],$$

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- ▶ That is, every n -point metric space admits a randomized partitioning algorithm so that the separation probabilities between pairs of points are **proportional** to the distances, up to a $\Theta(\log n)$ factor.
- ▶ The $\Theta(\log n)$ approximation factor is the best possible for **arbitrary** pseudometric spaces.

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Proof (Proposition & Bourgain's Theorem \implies Lemma).

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- ▶ By Proposition, \hat{y} is a pseudometric.
- ▶ By Bourgain's Theorem, there is a randomized algorithm that outputs a partition (A, B) and $\sigma > 0$ such that

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► **Exercise:** Prove Proposition and Bourgain's Theorem (Homework #4).

Dimensionality Reduction

Definition (α -Embeddings)

Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that $\phi : X \rightarrow Y$ is an α -**embedding** if there exists $r > 0$ such that

$$r \cdot d_X(u, v) \leq d_Y(\phi(u), \phi(v)) \leq r \cdot \alpha \cdot d_X(u, v).$$

for all $u, v \in X$.

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- ▶ **Dimensionality reduction** is the process of mapping a high dimensional dataset to a lower dimensional space, while preserving much of the important structure.
- ▶ For instance, let $X \subseteq \mathbb{R}^d$ and $Y = \mathbb{R}^t$ with $t < d$ and d_X, d_Y being the Euclidean distance.

Dimensionality Reduction

Theorem (Johnson-Lindenstrauss, 1984)

Let $x_1, \dots, x_n \in \mathbb{R}^d$. Let $\epsilon \in (0, 1)$. Then for some $t = O\left(\frac{\log(n)}{\epsilon^2}\right)$, there exist $y_1, \dots, y_n \in \mathbb{R}^t$ such that

$$\begin{aligned} (1 - \epsilon) \|x_j\| &\leq \|y_j\| \leq (1 + \epsilon) \|x_j\|, & \forall j \in [n], \\ (1 - \epsilon) \|x_j - x_{j'}\| &\leq \|y_j - y_{j'}\| \leq (1 + \epsilon) \|x_j - x_{j'}\|, & \forall j, j' \in [n]. \end{aligned}$$

Notation: $\|v\| = \sqrt{\sum_{i=1}^n v_i^2}$.

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Let $x_1, \dots, x_n \in \mathbb{R}^d$. Let $\epsilon \in (0, 1)$. Then for some $t = O(\frac{\log(n)}{\epsilon^2})$, there exist $y_1, \dots, y_n \in \mathbb{R}^t$ such that

$$\begin{aligned} (1 - \epsilon) \|x_j\| &\leq \|y_j\| \leq (1 + \epsilon) \|x_j\|, & \forall j \in [n], \\ (1 - \epsilon) \|x_j - x_{j'}\| &\leq \|y_j - y_{j'}\| \leq (1 + \epsilon) \|x_j - x_{j'}\|, & \forall j, j' \in [n]. \end{aligned}$$

Notation: $\|v\| = \sqrt{\sum_{i=1}^n v_i^2}$.

- ▶ **Remark:** There is a **random linear map** such that for any x_1, \dots, x_n the above condition holds with probability at least $\frac{1}{2n}$. This linear map is **oblivious**: it does not depend on x_1, \dots, x_n at all! In fact, the linear map is just a matrix whose entries are independent Gaussians.

Bourgain's Theorem & SPARSEST CUT

Theorem (Bourgain's Metric Embedding Theorem)

For any metric space (V, d) , there exists an $O(\log n)$ -embedding into $\mathbb{R}^{O(\log^2 n)}$ with the ℓ_1 -norm that is computable with high probability by a randomized polynomial-time algorithm.

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- ▶ This result is the best possible; i.e., there exists a metric that cannot be embedded into ℓ_1 with distortion less than $\Omega(\log n)$.

Bourgain's Theorem & SPARSEST CUT

Definition (Cut Metrics)

A metric (X, d) is a **cut metric** if there exists $S \subseteq X$ such that $d(x, y) = 0$ whenever $x, y \in S$ or $x, y \notin S$, and $d(x, y) = 1$ otherwise.

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A metric (X, d) is an ℓ_1 metric if and only if it is a nonnegative linear combination of cut metrics.

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- ▶ Lemma and Bourgain's Metric Embedding Theorem imply Bourgain's Theorem in the proof of the main theorem.

Bourgain's Theorem & SPARSEST CUT

Proof.

- ▶ (\Leftarrow) Let (X, d) be a nonnegative linear combination of cut metrics, i.e. $d = \sum_{k=1}^m \lambda_k \delta_{S_k}$ where $\lambda_k \geq 0$ and $S_k \subseteq X$.

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- ▶ Without loss of generality, $v_1 \leq \dots \leq v_n$.
- ▶ Then $d = \sum_{k=1}^{n-1} (v_{k+1} - v_k) \delta_{[k]}$.



Bourgain's Theorem & SPARSEST CUT

Problem (SPARSEST CUT)

Input: An undirected graph $G = (V, E)$ with edge weights $w_e > 0$ for each $e \in E$, and k pairs of vertices (s_i, t_i) each with demand d_i .

Goal: A set of vertices S that minimizes

$$\rho(S) \equiv \frac{\sum_{e \in \delta(S)} c_e}{\sum_{i: |S \cap \{s_i, t_i\}| = 1} d_i}.$$

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Corollary

There is a randomized $O(\log n)$ -approximation algorithm for SPARSEST CUT.

Tree Metric Embedding

Definition (Tree Metrics)

A metric (X, d) is a **tree metric** if there exists a tree $T = (V, E)$ with edge costs c_e for each $e \in E$ such that $d(u, v)$ is the cost of the unique path from u to v in T .

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Theorem (Fakcharoenphol-Rao-Talwar)

For any metric (V, d) such that $d(u, v) \geq 1$ for all $u, v \in V$ with $u \neq v$, there exists a randomized, polynomial-time algorithm that produces a tree metric (V', T) , $V \subseteq V'$ such that for all $u, v \in V$, we have $d(u, v) \leq T_{uv}$ and $\mathbb{E}[T_{uv}] \leq O(\log n)d(u, v)$.

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- ▶ The above result is obtained via the method of **hierarchical tree decomposition**.

Semidefinite Programming

Definition (Positive Semidefinite Matrices)

A matrix $X \in \mathbb{R}^{n \times n}$ is **positive semidefinite** (or **psd**), denoted $X \succeq 0$, if $y^T X y \geq 0$ for all $y \in \mathbb{R}^n$.

Fact

If $X \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then the following statements are equivalent:

1. X is psd;
2. X has nonnegative eigenvalues;
3. $X = V^T V$ for some $V \in \mathbb{R}^{m \times n}$ where $m \leq n$;
4. $X = \sum_{i=1}^n \lambda_i w_i w_i^T$ for some $\lambda_i \geq 0$ and $w_i \in \mathbb{R}^n$ such that $w_i^T w_i = 1$ and $w_i^T w_j = 0$ for all $i \neq j$.

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Definition (Semidefinite Programming, SDP)

A **semidefinite program**, or **SDP**, is a mathematical program with real-valued variables, a linear objective function, linear constraints, and a square symmetric matrix of variables constrained to be psd.

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- ▶ Below is an example of SDP with variables x_{ij} for $i, j \in [n]$:

$$\begin{aligned} \text{max or min} \quad & \sum_{i,j \in [n]} c_{ij} x_{ij} & (1) \\ \text{s.t.} \quad & \sum_{i,j \in [n]} a_{ijk} x_{ij} = b_k, & \forall k \in [n], \\ & x_{ij} = x_{ji}, & \forall i, j \in [n], \\ & X = (x_{ij}) \succeq 0. \end{aligned}$$

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- ▶ SDP can be solved to within an additive error of ϵ in polynomial time in the size of the input and $\log(\frac{1}{\epsilon})$.

Equivalence of SDP and Vector Programming

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A **vector program** is a mathematical program with variables $v_i \in \mathbb{R}^n$, where n is the number of vectors, and an objective function and constraints linear in the inner products of the vectors.

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$$\begin{aligned} \max \text{ or } \min \quad & \sum_{i,j \in [n]} c_{ij} (v_i \cdot v_j) & (2) \\ \text{s.t.} \quad & \sum_{i,j \in [n]} a_{ijk} (v_i \cdot v_j) = b_k, & \forall k \in [n], \\ & v_i \in \mathbb{R}^n, & \forall i \in [n]. \end{aligned}$$

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- ▶ Equivalently, $(z_i - z_j)^2$ is 0 if i, j are on the same side of the cut, and 4 otherwise.

Quadratic Programming Formulation

- ▶ Hence we obtain the **exact** quadratic programming formulation of MAXIMUM CUT:

$$\begin{aligned} \max \quad & \frac{1}{4} \sum_{ij \in E} w_{ij} (z_i - z_j)^2 \\ \text{s.t.} \quad & z_i \in \{-1, +1\}, \quad \forall i \in V. \end{aligned}$$

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- ▶ This quadratic program is **equivalent** to MAXIMUM CUT. Hence optimizing this program is *NP*-hard.

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- ▶ This vector program is a relaxation of the quadratic program by setting $v_i = (z_i, 0, \dots, 0) \in \mathbb{R}^n$.

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There is a randomized .878-approximation algorithm for MAXIMUM CUT.

Theorem

Given the unique game conjecture there is no α -approximation for MAXIMUM CUT with constant $\alpha > .878$ unless $P = NP$.

Vector Programming Relaxation

- ▶ For our purposes we want the vector programming relaxation to generalize the LP relaxation. Hence we want the analogs of the following two sets of constraints:

$$\begin{aligned}x_{ij} + x_{ik} + x_{jk} &\leq 2, & \forall i, j, k \in V \text{ distinct,} \\x_{jk} &\leq x_{ij} + x_{ik}, & \forall i, j, k \in V \text{ distinct.}\end{aligned}$$

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- ▶ This implies the following two sets of constraints in the form of inner products:

$$\begin{aligned}\|v_i - v_j\|^2 + \|v_i - v_k\|^2 + \|v_j - v_k\|^2 &\leq 8, & \forall i, j, k \in V \text{ distinct,} \\ \|v_j - v_k\|^2 &\leq \|v_i - v_j\|^2 + \|v_i - v_k\|^2, & \forall i, j, k \in V \text{ distinct.}\end{aligned}$$

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- ▶ The extended vector program, which we call (VP-MAXCUT), is still a relaxation for MAXIMUM CUT by setting v_i to $\pm e_1$ according to i 's side.

Improved Exact Recovery, Roadmap

- ▶ Recall the roadmap for the proof of the exact recovery for **MAXIMUM CUT** with $\gamma = \Theta(\log n)$.

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$$\hat{y}_{ij} = \begin{cases} \hat{x}_{ij}, & \text{if } i, j \text{ are on the same side of } C, \\ 1 - \hat{x}_{ij}, & \text{if } i, j \text{ are on different sides of } C. \end{cases}$$

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- ▶ This proposition and Bourgain's Theorem imply the lemma for the existence of randomized rounding algorithm that outputs a (random) cut such that the probability of an edge being cut is approximately the same as the value of the corresponding decision variable.
- ▶ The lemma then implies the exact recovery theorem by a common pattern used in perturbation-stable MINIMUM CUT and MINIMUM MULTIWAY CUT.

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- ▶ Fix an instance of MAXIMUM CUT. Let \hat{v}_i 's be an optimal solution to the vector program. Let $\hat{x}_{ij} = \frac{1}{4}\|v_i - v_j\|^2$ for any distinct $i, j \in V$.

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- ▶ Furthermore, \hat{x}_{ij} represents the squared Euclidean distances between points in \mathbb{R}^k for some k .

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- ▶ Fix an instance of MAXIMUM CUT. Let \hat{v}_i 's be an optimal solution to the vector program. Let $\hat{x}_{ij} = \frac{1}{4} \|v_i - v_j\|^2$ for any distinct $i, j \in V$.
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Definition (ℓ_2^2 Metrics)

A metric (X, d) is an ℓ_2^2 **metric** if it represents squared Euclidean distances between points in \mathbb{R}^k for some k .

That is, there exists an embedding from X to \mathbb{R}^k for some k such that $d(\cdot, \cdot)$'s are squared Euclidean distances between points in \mathbb{R}^k .

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Theorem (Arora, Lee, and Naor, 2005)

For every n -point ℓ_2^2 metric space (X, d) , there exists a randomized algorithm that generates a random partition (A, B) of X and a scaling parameter $\sigma > 0$ such that, for all distinct $i, j \in X$,

$$\mathbb{P}[i, j \text{ on different sides of } (A, B)] \in \sigma \cdot \left[\frac{d(i, j)}{\alpha}, d(i, j) \right],$$

where $\alpha = O(\sqrt{\log n \log \log n})$.

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- ▶ Arora, Lee, and Naor's Theorem gives the lower bound for the probability of an edge being cut. Similarly, we need an upper bound for the probability of an edge not being cut.

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Proposition

Fix an instance of MAXIMUM CUT, a cut C , and a feasible solution \hat{v}_i 's to (VP-MAXCUT). Let \hat{x} be the induced ℓ_2^2 metric $\hat{x}_{ij} = \frac{1}{4} \|v_i - v_j\|^2$. For distinct $i, j \in V$, define

$$\hat{y}_{ij} = \begin{cases} \hat{x}_{ij}, & \text{if } i, j \text{ are on the same side of } C, \\ 1 - \hat{x}_{ij}, & \text{if } i, j \text{ are on different sides of } C. \end{cases}$$

Then \hat{y} is also an ℓ_2^2 metric.

Improved Exact Recovery, Roadmap

Proof.

- ▶ \hat{y} is a metric by the triangle inequality constraints of (VP-MAXCUT).

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- ▶ \hat{y} is a metric by the triangle inequality constraints of (VP-MAXCUT).
- ▶ To prove that \hat{y} is an ℓ_2^2 metric, we need to exhibit an embedding of V into \mathbb{R}^k for some k such that \hat{y}_{ij} 's are squared Euclidean distances between points in \mathbb{R}^k .

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$$\hat{u}_i = \begin{cases} \hat{v}_i, & \text{if } i \in A, \\ -\hat{v}_i, & \text{if } i \in B. \end{cases}$$

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- ▶ Then $\hat{y}_{ij} = \frac{1}{4} \|\hat{u}_i - \hat{u}_j\|^2$ for any distinct $i, j \in V$.



Improved Exact Recovery, Roadmap

Lemma

Fix an instance of MAXIMUM CUT, with F^* the edges in the optimal cut, and v_i 's the optimal solution to (VP-MAXCUT). Then there exists a randomized algorithm that generates a random cut (A, B) and a scaling parameter $\sigma > 0$ such that:

1. For every edge $e = ij \notin F^*$,

$$\mathbb{P}[e \text{ cut by } (A, B)] \geq \sigma \cdot \frac{\frac{1}{4} \|v_i - v_j\|^2}{\alpha},$$

where $\alpha = \Theta(\sqrt{\log n \log \log n})$;

2. For every edge $e = ij \in F^*$,

$$\mathbb{P}[e \text{ not cut by } (A, B)] \leq \sigma \cdot \left(1 - \frac{1}{4} \|v_i - v_j\|\right);$$

3. The rounding algorithm is deterministic iff. \hat{v}_i 's are integral.

Improved Exact Recovery, Roadmap

- ▶ By \hat{v}_i 's being **integral**, we mean that there exist antipodal unit vectors $w, -w$ such that $\hat{v}_i \in \{w, -w\}$ for each $i \in V$.

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Theorem

There is a constant $c > 0$ such that in every $(c\sqrt{\log n \log \log n})$ -perturbation-stable instance of MAXIMUM CUT with n vertices, every optimal solution to (VP-MAXCUT) is integral.

Can We Do Better?

Theorem

There exist $O(\sqrt{\log n})$ -perturbation-stable instances of MAXIMUM CUT for which the optimal solution to (VP-MAXCUT) is not integral.

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Assuming the Unique Game Conjecture, for every constant $\gamma \geq 1$, there is no polynomial-time algorithm for certifiable exact recovery in γ -perturbation-stable instances of MAXIMUM CUT.

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Assuming the Unique Game Conjecture, for every constant $\gamma \geq 1$, there is no polynomial-time algorithm for certifiable exact recovery in γ -perturbation-stable instances of MAXIMUM CUT.

- ▶ Both results are based upon a reduction from SPARSEST CUT to MAXIMUM CUT. See *Bilu–Linial Stable Instances of Max Cut and Minimum Multiway Cut* by Makarychev, Makarychev, and Vijayaraghavan (2013) and Homework #4.