Summary: *Roundtrip Spanners and Roundtrip Routing in Directed Graphs*

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In lectures of 6.890, we have seen the notion of spanners of an *undirected* graph. We say that a subgraph H of a weighted undirected graph $G = (V, E)$ is a *t-spanner* of G if $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$, where $d(u, v)$ is the distance between u and v in the graph denoted by the subscript. This definition, however, is not interesting in the *directed* setting, as directed graphs do not contain sparse spanners in general. To see this, consider a directed bipartite graph G, in which case removing any edge (u, v) from G makes the distance between u and v infinite in the resulting subgraph, so G does not admit a sparse spanner.

Cowen and Wagner [3, 4] introduced the notion of *roundtrip spanners* to remedy the directed case. We say that a subgraph H of a weighted directed graph $G = (V, E)$ is a *t-roundtrip-spanner* of G if $d_H(u \rightleftarrows v) \leq t \cdot d_G(u \rightleftarrows v)$ for all $u, v \in V$, where $d(u \rightleftarrows v) = d(u \to v) + d(v \to u)$ is the *roundtrip distance* between u and v in the graph denoted by the subscript. We observe that $d_G(u \rightleftarrows v) = 2d_G(u, v)$ for an undirected graph G, so roundtrip spanners of undirected graphs are simply conventional spanners.

One of the main contributions of Roditty et al. [6] is the following theorem:

Theorem 1 (Roditty et al. [6]). *For all $k \in \mathbb{N}$ and $\varepsilon > 0$, every weighted directed graph on n vertices with edge weights from $[1, W]$ has a $(2k + \varepsilon)$-roundtrip-spanner with $O((k^2/\varepsilon)n^{1+1/k} \log(nW))$ edges.*

Theorem 1 is an exponential improvement of [4], which, given a weighted directed graph, asserts the existence of a $(2^k - 1)$-roundtrip-spanner with $\tilde{O}(n^{1+1/k})$ edges for all $k \in \mathbb{N}$. The proof of Theorem 1 in [6] adapts a technique introduced by Cohen [2] for undirected graphs. Let $G = (V, E)$ be a weighted directed graph on n vertices with edge weights from $[1, W]$. Let $\varepsilon > 0$. For each $i \in [\log_{1+\varepsilon}(2nW)]$, we construct a subgraph H of G with $O(kn^{1+1/k})$ edges such that $d_H(u \rightleftarrows v) \leq 2kR_i$ for all $u, v \in V$ with $d_G(u \rightleftarrows v) \leq R_i$, where $R_i = (1 + \varepsilon)^i$. It follows that the union of all these subgraphs is a $2k(1 + \varepsilon)$-roundtrip-spanner of G with $O(kn^{1+1/k} \log_{1+\varepsilon}(nW))$ edges.

The construction of each such subgraph H relies upon the notions of *balls* and *(k, R)-roundtrip-covers*. For each $V' \subset V$, $v \in V$ and $r \geq 0$, the *ball* centered at v of radius r, denoted by $\mathrm{ball}_{V'}(v, r)$, is defined to be the set of vertices at roundtrip distance at most r from v in $G[V']$. Moreover, a *(k, R)-roundtrip-cover* is a collection $\mathcal{C}$ of balls of radius at most kR such that, for all $u, v \in V$ with $d_G(u \rightleftarrows v) \leq R$, there exists a ball $B \in \mathcal{C}$ such that $u, v \in B$. Roditty et al. [6] gives an iterative sampling procedure for constructing a (k, R)-roundtrip-cover, by adding to the cover $\mathrm{ball}_{V_i}(v, (i + 1)R)$ for each sampled vertex v from V_i and removing the corresponding *core* $\mathrm{ball}_{V_i}(v, iR)$ from the graph, in iterations $i = k - 1, \ldots, 0$. By setting the sampling ratio to $n^{-i/k}$ in each iteration i, Roditty et al. [6] shows that the *expected* number of balls in the produced (k, R)-roundtrip-cover is at most $kn^{1/k}$, adapting an argument similar to the one in the $(2k - 1)$-approximate distance oracle of Thorup and Zwick [8] (see also Lecture 15 of 6.890).

The key graph-theoretic structure in the proof of Theorem 1 is the notions of *in- and out-trees*. For each $V' \subset V$, $v \in V$ and $r \geq 0$, we define $\mathrm{OutTree}(B, v)$ to be a tree containing directed shortest paths from v to all vertices in $B := \mathrm{ball}_{V'}(v, r)$ in $G[V']$, and similarly, define $\mathrm{InTree}(B, v)$ to be a tree containing directed shortest paths from all vertices in B to v in $G[V']$. Moreover, we define $\mathrm{InOutTrees}(B, v) = \mathrm{InTree}(B, V) \cup \mathrm{OutTree}(B, v)$, called a *double-tree*.

Double-trees have nice structural properties which can be used to design sparse roundtrip spanners and compact routing schemes, as discussed in [6]. Specifically, we have the following two simple lemmas:

Lemma 2 (Roditty et al. [6]). *For each $V' \subset V$, $v \in V$ and $r \geq 0$, if $u \in B := \mathrm{ball}_{V'}(v, r)$ and if w is on a shortest path from u to v, or from v to u, in $G[V']$, then $w \in B$. Hence, $V(\mathrm{InTree}(B, v)), V(\mathrm{OutTree}(B, v)) \subset B$, so $V(\mathrm{InOutTrees}(B, v)) \subset B$. It follows that $|E(\mathrm{InOutTrees}(B, v))| \leq 2(|B| - 1)$.*

Lemma 3 (Roditty et al. [6]). *For each $V' \subset V$, $v \in V$ and $r \geq 0$, if $u_1, u_2 \in B := \mathrm{ball}_{V'}(v, r)$, then $\mathrm{InOutTrees}(B, v)$ contains a closed directed tour containing u_1 and u_2 of length at most $2r$.*

Combining Lemmas 2 and 3, we can prove Theorem 1 by iteratively considering roundtrip distances that are at most R_i, as mentioned in a previous paragraph. Although we do not discuss routing schemes in this summary, we note that nice properties of double-trees also allow us to design compact routing schemes in directed graphs, as an adaptation of the construction reduces roundtrip routing in directed graphs to routing in double-trees, which is discussed in, e.g., Fraigniaud and Gavoille [5] and Thorup and Zwick [7].

To conclude this summary, we note that any undirected graph contains a $(2k - 1)$-spanner with $O(n^{1+1/k})$ edges, which is believed to be tight assuming the Erdős girth conjecture (see, e.g., Lecture 14 of 6.890). Roditty et al. [6] asks whether their $(2k + \varepsilon)$-roundtrip-spanner can be improved to a $(2k - 1)$-roundtrip-spanner. This question was recently answered by Cen et al. [1], who gave a deterministic algorithm for constructing a $(2k - 1)$-roundtrip-spanner with $O(kn^{1+1/k} \log n)$ edges for all $k \in \mathbb{N}$.

References

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