On the Sensitivity of Boolean Functions

Yuchong Pan

UBC CPSC 531F

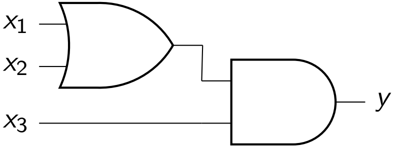
April 13, 2021

Paul Erdős famously spoke of a book, maintained by God, in which was written the simplest, most beautiful proof of each theorem. The highest compliment Erdős could give a proof was that it "came straight from the book." In this case, I find it hard to imagine that even God knows how to prove the Sensitivity Conjecture in any simpler way than this.

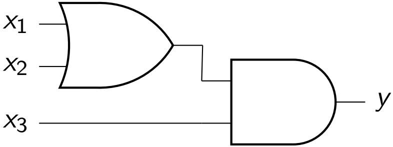
— *Scott Aaronson*[1]

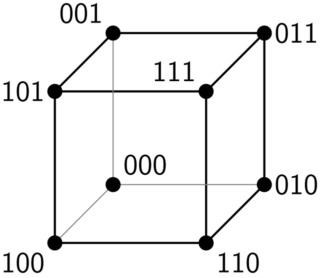
[1] https://www.scottaaronson.com/blog/?p=4229.

Sensitivity of Boolean Functions

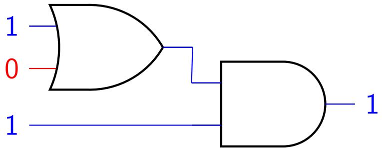


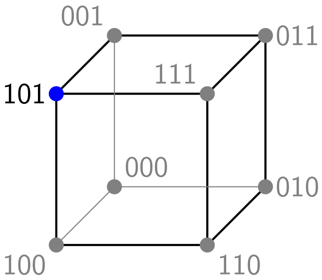
Sensitivity of Boolean Functions



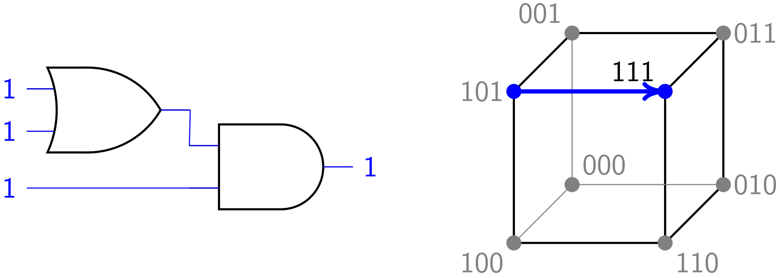


Sensitivity of Boolean Functions

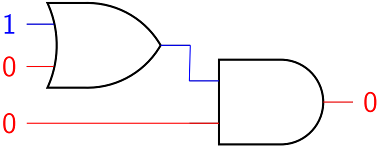


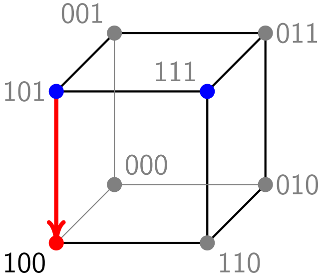


Sensitivity of Boolean Functions

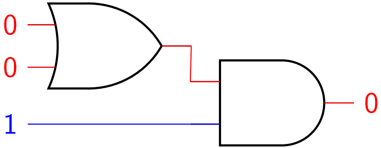


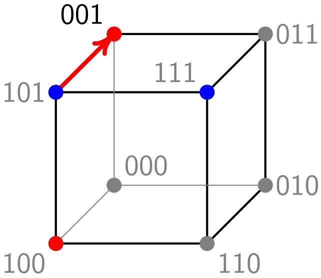
Sensitivity of Boolean Functions



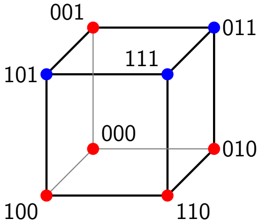


Sensitivity of Boolean Functions

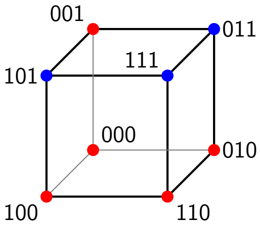




Sensitivity of Boolean Functions



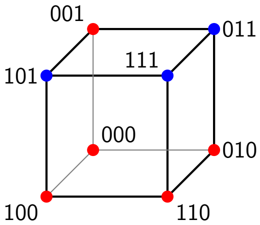
Sensitivity of Boolean Functions



Definition

For $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by x^S the binary vector obtained from x by flipping indices from S.

Sensitivity of Boolean Functions



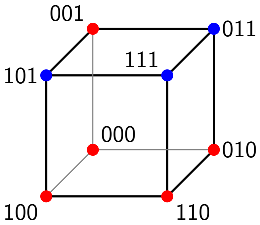
Definition

For $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by x^S the binary vector obtained from x by flipping indices from S.

Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local sensitivity** $s(f,x)$ of f on the input x is defined as the number of indices i such that $f(x) \neq f(x^{\{i\}})$.

Sensitivity of Boolean Functions



Definition

For $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by x^S the binary vector obtained from x by flipping indices from S.

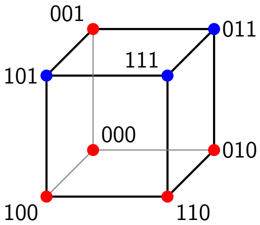
Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local sensitivity** $s(f,x)$ of f on the input x is defined as the number of indices i such that $f(x) \neq f(x^{\{i\}})$.

Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **sensitivity** $s(f)$ of f is defined as $\max_{x \in \{0,1\}^n} s(f,x)$.

Sensitivity of Boolean Functions



$s(f, 101) = 2$

Definition

For $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by x^S the binary vector obtained from x by flipping indices from S.

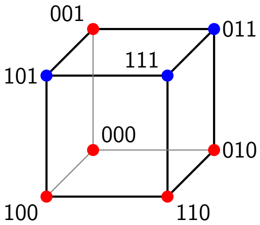
Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local sensitivity** $s(f,x)$ of f on the input x is defined as the number of indices i such that $f(x) \neq f(x^{\{i\}})$.

Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **sensitivity** $s(f)$ of f is defined as $\max_{x \in \{0,1\}^n} s(f,x)$.

Sensitivity of Boolean Functions



$$s(f, 101) = 2$$
$$s(f) = 2$$

Definition

For $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by x^S the binary vector obtained from x by flipping indices from S.

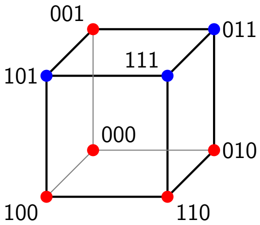
Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local sensitivity** $s(f,x)$ of f on the input x is defined as the number of indices i such that $f(x) \neq f(x^{\{i\}})$.

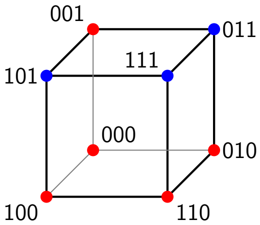
Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **sensitivity** $s(f)$ of f is defined as $\max_{x \in \{0,1\}^n} s(f,x)$.

Block Sensitivity of Boolean Functions



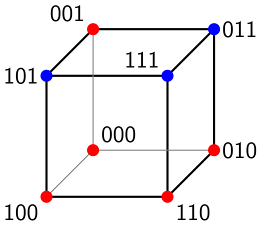
Block Sensitivity of Boolean Functions



Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local block sensitivity** $bs(f, x)$ of f on the input x is defined as the maximum number of disjoint blocks $B_1, \ldots, B_k$ of $[n]$ such that $f(x) \neq f(x^{B_i})$ for each $i \in [k]$.

Block Sensitivity of Boolean Functions



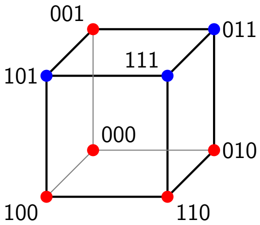
Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local block sensitivity** $bs(f,x)$ of f on the input x is defined as the maximum number of disjoint blocks $B_1, \ldots, B_k$ of $[n]$ such that $f(x) \neq f(x^{B_i})$ for each $i \in [k]$.

Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **block sensitivity** $bs(f)$ of f is defined as $\max_{x \in \{0,1\}^n} bs(f,x)$.

Block Sensitivity of Boolean Functions



Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local block sensitivity** $bs(f,x)$ of f on the input x is defined as the maximum number of disjoint blocks $B_1, \ldots, B_k$ of $[n]$ such that $f(x) \neq f(x^{B_i})$ for each $i \in [k]$.

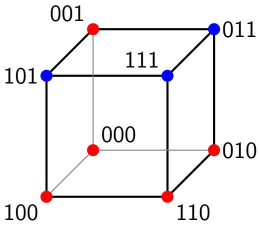
Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **block sensitivity** $bs(f)$ of f is defined as $\max_{x \in \{0,1\}^n} bs(f,x)$.

Observation

For $f : \{0,1\}^n \to \{0,1\}$, $bs(f) \geq s(f)$.

Block Sensitivity of Boolean Functions



$bs(f, 101) = 2$

Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local block sensitivity** $bs(f,x)$ of f on the input x is defined as the maximum number of disjoint blocks $B_1, \ldots, B_k$ of $[n]$ such that $f(x) \neq f(x^{B_i})$ for each $i \in [k]$.

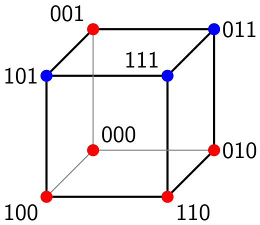
Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **block sensitivity** $bs(f)$ of f is defined as $\max_{x \in \{0,1\}^n} bs(f,x)$.

Observation

For $f : \{0,1\}^n \to \{0,1\}$, $bs(f) \geq s(f)$.

Block Sensitivity of Boolean Functions



$bs(f, 101) = 2$

$bs(f) = 2$

Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **local block sensitivity** $bs(f,x)$ of f on the input x is defined as the maximum number of disjoint blocks $B_1, \ldots, B_k$ of $[n]$ such that $f(x) \neq f(x^{B_i})$ for each $i \in [k]$.

Definition

For $f : \{0,1\}^n \to \{0,1\}$, the **block sensitivity** $bs(f)$ of f is defined as $\max_{x \in \{0,1\}^n} bs(f,x)$.

Observation

For $f : \{0,1\}^n \to \{0,1\}$, $bs(f) \geq s(f)$.

Sensitivity vs. Block Sensitivity

Question 1

Is it possible that $bs(f) > s(f)$?

Sensitivity vs. Block Sensitivity

Question 1

Is it possible that $bs(f) > s(f)$?

- Yes! Let $f : \{0,1\}^8 \to \{0,1\}$ be such that $f(x) = 1$ if and only if the number of 1's in x is either 4 or 5.

Sensitivity vs. Block Sensitivity

Question 1

Is it possible that $bs(f) > s(f)$?

- Yes! Let $f : \{0,1\}^8 \to \{0,1\}$ be such that $f(x) = 1$ if and only if the number of 1's in x is either 4 or 5.
- For most inputs x, e.g., $x = 00000000$, $s(f,x) = 0$.

Sensitivity vs. Block Sensitivity

Question 1

Is it possible that $bs(f) > s(f)$?

- Yes! Let $f : \{0,1\}^8 \to \{0,1\}$ be such that $f(x) = 1$ if and only if the number of 1's in x is either 4 or 5.
- For most inputs x, e.g., $x = 00000000$, $s(f,x) = 0$.
- For $x = 11100000$, $f(x^{\{i\}}) \neq f(x)$ for $i \in \{4,\ldots,8\}$, so $s(f,x) = 5$. Indeed, $s(f) = 5$.

Sensitivity vs. Block Sensitivity

Question 1

Is it possible that $bs(f) > s(f)$?

- Yes! Let $f : \{0,1\}^8 \to \{0,1\}$ be such that $f(x) = 1$ if and only if the number of 1's in x is either 4 or 5.
- For most inputs x, e.g., $x = 00000000$, $s(f,x) = 0$.
- For $x = 11100000$, $f(x^{\{i\}}) \neq f(x)$ for $i \in \{4, \ldots, 8\}$, so $s(f,x) = 5$. Indeed, $s(f) = 5$.
- For $x = 11110000$, then $\{1\}, \{2\}, \{3\}, \{4\}, \{5,6\}, \{7,8\}$ are 6 disjoint, **sensitive** blocks for f, so $bs(f) \geq 6 > s(f)$.

Sensitivity vs. Block Sensitivity

Question 2

How large can $bs(f)$ be compared to $s(f)$, asymptotically?

Sensitivity vs. Block Sensitivity

Question 2

How large can $bs(f)$ be compared to $s(f)$, asymptotically?

Theorem (Rubinstein 1995)

There exists an infinite family of Boolean functions f such that

$$bs(f) = \Omega\left(s(f)^2\right).$$

Sensitivity vs. Block Sensitivity

Question 2

How large can $bs(f)$ be compared to $s(f)$, asymptotically?

Theorem (Rubinstein 1995)

There exists an infinite family of Boolean functions f such that

$$bs(f) = \Omega\left(s(f)^2\right).$$

Define $f : \{0,1\}^{n^2} \to \{0,1\}$ as

$$f\left(x_{11}, \ldots, x_{nn}\right) = \bigvee_{i=1}^{n} g\left(x_{i1}, \ldots, x_{in}\right),$$

where $g(x_1, \ldots, x_n)$ if and only if $x_j = x_{j+1} = 1$ for some $j \in [n-1]$, and all other $x_k = 0$.

Rubinstein's Function

Claim

$bs(f) \geq bs(f, 0) = \Omega(n^2)$.

Proof.

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}
$$

□

Rubinstein's Function

Claim

$bs(f) \geq bs(f,0) = \Omega(n^2)$.

Proof.

$$
\begin{array}{cccccccc}
\color{red}{1} & \color{red}{1} & 0 & 0 & \cdots & \rightarrow & \color{red}{1} \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots \\
 & & & & & & \downarrow \\
 & & & & & & \color{red}{1}
\end{array}
$$

□

Rubinstein's Function

Claim

$bs(f) \geq bs(f,0) = \Omega(n^2)$.

Proof.

$$
\begin{array}{cccccccc}
0 & 0 & \color{red}{1} & \color{red}{1} & \cdots & \rightarrow & \color{red}{1} \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots \\
 & & & & & & \downarrow \\
 & & & & & & \color{red}{1}
\end{array}
$$

$\square$

Rubinstein's Function

Claim

$s(f) = O(n)$.

Proof.

- **Case 1:** $f(x) = 0$.
  Each row must output 0.

Rubinstein's Function

Claim

$s(f) = O(n)$.

Proof.

- **Case 1:** $f(x) = 0$.
  Each row must output 0.
  There are at most two **sensitive** indices on each row, e.g.,

$$0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0$$

Rubinstein's Function

Claim

$s(f) = O(n)$.

Proof.

- **Case 1:** $f(x) = 0$.
  Each row must output 0.
  There are at most two **sensitive** indices on each row, e.g.,

  $$0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0$$

  Hence, $s(f, x) \leq 2n$.

Rubinstein's Function

Claim

$s(f) = O(n)$.

Proof.

- **Case 1:** $f(x) = 0$.
  Each row must output 0.
  There are at most two **sensitive** indices on each row, e.g.,

  $$0 \quad \dots \quad 0 \quad 1 \quad 0 \quad \dots \quad 0$$

  Hence, $s(f, x) \leq 2n$.
- **Case 2:** $f(x) = 1$.

Rubinstein's Function

Claim

$s(f) = O(n)$.

Proof.

- **Case 1:** $f(x) = 0$.
  Each row must output 0.
  There are at most two **sensitive** indices on each row, e.g.,
  $$0 \quad \ldots \quad 0 \quad 1 \quad 0 \quad \ldots \quad 0$$
  Hence, $s(f, x) \leq 2n$.
- **Case 2:** $f(x) = 1$.
  - If two rows output 1, $s(f, x) = 0$.

Rubinstein's Function

Claim

$s(f) = O(n)$.

Proof.

- **Case 1:** $f(x) = 0$.
  Each row must output 0.
  There are at most two **sensitive** indices on each row, e.g.,

  $$0 \quad \ldots \quad 0 \quad 1 \quad 0 \quad \ldots \quad 0$$

  Hence, $s(f, x) \leq 2n$.
- **Case 2:** $f(x) = 1$.
  - If two rows output 1, $s(f, x) = 0$.
  - If only one row outputs 1, $s(f, x) \leq n$.

□

Sensitivity vs. Block Sensitivity

Question 3 (Nisan and Szegedy 1992)

Is $bs(f)$ always polynomial in $s(f)$?

Sensitivity vs. Block Sensitivity

Question 3 (Nisan and Szegedy 1992)

Is $bs(f)$ always polynomial in $s(f)$?

Theorem (Huang 2019)

For every Boolean function f,

$$bs(f) \leq s(f)^4.$$

Why?

Why?

Definition

Complexity measures α, β of Boolean functions are **polynomially related** if there exist polynomials p_1, p_2 such that for every Boolean function f,

$$\alpha(f) \leq p_1(\beta(f)), \qquad \beta(f) \leq p_2(\alpha(f)).$$

Why?

Definition

Complexity measures α, β of Boolean functions are **polynomially related** if there exist polynomials p_1, p_2 such that for every Boolean function f,

$$\alpha(f) \leq p_1(\beta(f)), \qquad \beta(f) \leq p_2(\alpha(f)).$$

Theorem (Hatami, Kulkarni, and Pankratov 2010)

The following complexity measures are polynomially related:

- *block sensitivity*
- *decision tree complexity*
- *certificate complexity*
- *degree as polynomial*
- *approximate degree*
- *randomized query complexity*
- *quantum query complexity*

Why?

So if, as is conjectured, sensitivity and block-sensitivity are polynomially related, then sensitivity—arguably the most basic of all Boolean function complexity measures——ceases to be an outlier and joins a large and happy flock.

— *Scott Aaronson*[2]

[2] https://www.scottaaronson.com/blog/?p=453.

Why?

> *So if, as is conjectured, sensitivity and block-sensitivity are polynomially related, then sensitivity—arguably the most basic of all Boolean function complexity measures——ceases to be an outlier and joins a large and happy flock.*
>
> — *Scott Aaronson*[2]

- Low-sensitivity Boolean functions are easy to compute in computational models like the deterministic decision tree.

[2] `https://www.scottaaronson.com/blog/?p=453`.

Why?

> *So if, as is conjectured, sensitivity and block-sensitivity are polynomially related, then sensitivity—arguably the most basic of all Boolean function complexity measures——ceases to be an outlier and joins a large and happy flock.*
>
> — *Scott Aaronson*[2]

- Low-sensitivity Boolean functions are easy to compute in computational models like the deterministic decision tree.
- Low-sensitivity Boolean functions have low degrees as real polynomials.

[2] https://www.scottaaronson.com/blog/?p=453.

Why?

> *So if, as is conjectured, sensitivity and block-sensitivity are polynomially related, then sensitivity—arguably the most basic of all Boolean function complexity measures——ceases to be an outlier and joins a large and happy flock.*
>
> — *Scott Aaronson*[2]

- Low-sensitivity Boolean functions are easy to compute in computational models like the deterministic decision tree.
- Low-sensitivity Boolean functions have low degrees as real polynomials.
- Any randomized algorithm to guess the parity of an n-bit string, which succeeds with probability $\geq \frac{2}{3}$ on the majority of strings, must make at least $\sim \sqrt{n}$ queries to the string, while any such quantum algorithm must make at least $\sim n^{1/4}$ queries (Aaronson et al. 2014).

[2] `https://www.scottaaronson.com/blog/?p=453.`

Notations

Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.

Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.
- For an undirected graph G, we use $\Delta(G)$ to denote the maximum degree of G.

Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.
- For an undirected graph G, we use $\Delta(G)$ to denote the maximum degree of G.
- For an undirected graph G and an induced subgraph H of G, we use $G \setminus H$ to denote the subgraph of G induced by the vertex set $V(G) \setminus V(H)$.

Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.
- For an undirected graph G, we use $\Delta(G)$ to denote the maximum degree of G.
- For an undirected graph G and an induced subgraph H of G, we use $G \setminus H$ to denote the subgraph of G induced by the vertex set $V(G) \setminus V(H)$.
- We use $\mathbb{B}^n$ to denote the n-dimensional Boolean hypercube.

Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.
- For an undirected graph G, we use $\Delta(G)$ to denote the maximum degree of G.
- For an undirected graph G and an induced subgraph H of G, we use $G \setminus H$ to denote the subgraph of G induced by the vertex set $V(G) \setminus V(H)$.
- We use $\mathbb{B}^n$ to denote the n-dimensional Boolean hypercube.
- For an induced subgraph H of $\mathbb{B}^n$, let

$$\Gamma(H) = \max\left\{\Delta(H), \Delta\left(\mathbb{B}^n \setminus H\right)\right\}.$$

Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.
- For an undirected graph G, we use $\Delta(G)$ to denote the maximum degree of G.
- For an undirected graph G and an induced subgraph H of G, we use $G \setminus H$ to denote the subgraph of G induced by the vertex set $V(G) \setminus V(H)$.
- We use $\mathbb{B}^n$ to denote the n-dimensional Boolean hypercube.
- For an induced subgraph H of $\mathbb{B}^n$, let

$$\Gamma(H) = \max\left\{\Delta(H), \Delta\left(\mathbb{B}^n \setminus H\right)\right\}.$$

- Given symmetric $A \in \mathcal{M}_m(\mathbb{R})$, let the real eigenvalues of A be ordered such that $\lambda_1(A) \geq \ldots \geq \lambda_m(A)$, counting multiplicity.

Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.
- For an undirected graph G, we use $\Delta(G)$ to denote the maximum degree of G.
- For an undirected graph G and an induced subgraph H of G, we use $G \setminus H$ to denote the subgraph of G induced by the vertex set $V(G) \setminus V(H)$.
- We use $\mathbb{B}^n$ to denote the n-dimensional Boolean hypercube.
- For an induced subgraph H of $\mathbb{B}^n$, let

$$\Gamma(H) = \max\left\{\Delta(H), \Delta\left(\mathbb{B}^n \setminus H\right)\right\}.$$

- Given symmetric $A \in \mathcal{M}_m(\mathbb{R})$, let the real eigenvalues of A be ordered such that $\lambda_1(A) \geq \ldots \geq \lambda_m(A)$, counting multiplicity.
- Given a Boolean function f, we use $\deg(f)$ to denote the **degree** of f, i.e., degree of the unique multilinear real polynomial that represents f.

Huang (2019)'s Roadmap

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, we have $\Gamma(H) \geq h(n)$.*
- *For any Boolean function f, we have $s(f) \geq h(\deg(f))$.*

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, we have $\Gamma(H) \geq h(n)$.*
- *For any Boolean function f, we have $s(f) \geq h(\deg(f))$.*

- **Idea 1: largest eigenvalue.** $\Delta(G) \geq \lambda_1(A)$ for a graph G, where A is a signed adjacency matrix of G.

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph* H *of* $\mathbb{B}^n$ *with* $|V(H)| \neq 2^{n-1}$*, we have* $\Gamma(H) \geq h(n)$.
- *For any Boolean function* f*, we have* $s(f) \geq h(\deg(f))$.

- **Idea 1: largest eigenvalue.** $\Delta(G) \geq \lambda_1(A)$ for a graph G, where A is a signed adjacency matrix of G.
- **Idea 2: eigenvalue interlace.** If B is a $k \times k$ principal submatrix of $A \in \mathcal{M}_m(\mathbb{R})$ symmetric, $\lambda_1(B) \geq \lambda_{m-k+1}(A)$.

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph* H *of* $\mathbb{B}^n$ *with* $|V(H)| \neq 2^{n-1}$, *we have* $\Gamma(H) \geq h(n)$.
- *For any Boolean function* f, *we have* $s(f) \geq h(\deg(f))$.

- **Idea 1: largest eigenvalue.** $\Delta(G) \geq \lambda_1(A)$ for a graph G, where A is a signed adjacency matrix of G.
- **Idea 2: eigenvalue interlace.** If B is a $k \times k$ principal submatrix of $A \in \mathcal{M}_m(\mathbb{R})$ symmetric, $\lambda_1(B) \geq \lambda_{m-k+1}(A)$.

$$m = 2^n, k = 2^{n-1} + 1 \implies \lambda_1(B) \geq \lambda_{2^{n-1}}(A).$$

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, we have $\Gamma(H) \geq h(n)$.*
- *For any Boolean function f, we have $s(f) \geq h(\deg(f))$.*

- **Idea 1: largest eigenvalue.** $\Delta(G) \geq \lambda_1(A)$ for a graph G, where A is a signed adjacency matrix of G.
- **Idea 2: eigenvalue interlace.** If B is a $k \times k$ principal submatrix of $A \in \mathcal{M}_m(\mathbb{R})$ symmetric, $\lambda_1(B) \geq \lambda_{m-k+1}(A)$.
$$m = 2^n, k = 2^{n-1} + 1 \implies \lambda_1(B) \geq \lambda_{2^{n-1}}(A).$$
- **Idea 3: magic signed adjacency matrix.** Constructs a signed adjacency matrix A of $\mathbb{B}^n$ with $\lambda_{2^{n-1}}(A) = \sqrt{n}$.

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph* H *of* $\mathbb{B}^n$ *with* $|V(H)| \neq 2^{n-1}$, *we have* $\Gamma(H) \geq h(n)$.
- *For any Boolean function* f, *we have* $s(f) \geq h(\deg(f))$.

- **Idea 1: largest eigenvalue.** $\Delta(G) \geq \lambda_1(A)$ for a graph G, where A is a signed adjacency matrix of G.
- **Idea 2: eigenvalue interlace.** If B is a $k \times k$ principal submatrix of $A \in \mathcal{M}_m(\mathbb{R})$ symmetric, $\lambda_1(B) \geq \lambda_{m-k+1}(A)$.
$$m = 2^n, k = 2^{n-1} + 1 \implies \lambda_1(B) \geq \lambda_{2^{n-1}}(A).$$
- **Idea 3: magic signed adjacency matrix.** Constructs a signed adjacency matrix A of $\mathbb{B}^n$ with $\lambda_{2^{n-1}}(A) = \sqrt{n}$.
- If H is an induced subgraph of $\mathbb{B}^n$ with $|V(H)| = 2^{n-1} + 1$,
$$\Delta(H) \geq \sqrt{n}.$$

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph* H *of* $\mathbb{B}^n$ *with* $|V(H)| \neq 2^{n-1}$, *we have* $\Gamma(H) \geq h(n)$.
- *For any Boolean function* f, *we have* $s(f) \geq h(\deg(f))$.

Theorem (Huang 2019)

For $n \in \mathbb{N}$, *if* H *is a* $(2^{n-1}+1)$*-vertex induced subgraph of* $\mathbb{B}^n$, *then*

$$\Delta(H) \geq \sqrt{n}.$$

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, we have $\Gamma(H) \geq h(n)$.*
- *For any Boolean function f, we have $s(f) \geq h(\deg(f))$.*

Theorem (Huang 2019)

For $n \in \mathbb{N}$, if H is a $(2^{n-1}+1)$-vertex induced subgraph of $\mathbb{B}^n$, then

$$\Delta(H) \geq \sqrt{n}.$$

- For any induced subgrpah H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, one of H and $\mathbb{B}^n \setminus H$ contains at least $2^{n-1}+1$ vertices.

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph* H *of* $\mathbb{B}^n$ *with* $|V(H)| \neq 2^{n-1}$, *we have* $\Gamma(H) \geq h(n)$.
- *For any Boolean function* f, *we have* $s(f) \geq h(\deg(f))$.

Theorem (Huang 2019)

For $n \in \mathbb{N}$, *if* H *is a* $(2^{n-1}+1)$*-vertex induced subgraph of* $\mathbb{B}^n$, *then*

$$\Delta(H) \geq \sqrt{n}.$$

- For any induced subgrpah H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, one of H and $\mathbb{B}^n \setminus H$ contains at least $2^{n-1}+1$ vertices.
- Gotsman and Linial (1992): $s(f) \geq \sqrt{\deg(f)}$.

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph* H *of* $\mathbb{B}^n$ *with* $|V(H)| \neq 2^{n-1}$, *we have* $\Gamma(H) \geq h(n)$.
- *For any Boolean function* f, *we have* $s(f) \geq h(\deg(f))$.

Theorem (Huang 2019)

For $n \in \mathbb{N}$, *if* H *is a* $(2^{n-1}+1)$*-vertex induced subgraph of* $\mathbb{B}^n$, *then*

$$\Delta(H) \geq \sqrt{n}.$$

- For any induced subgrpah H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, one of H and $\mathbb{B}^n \setminus H$ contains at least $2^{n-1}+1$ vertices.
- Gotsman and Linial (1992): $s(f) \geq \sqrt{\deg(f)}$.
- Tal (2013): $bs(f) \leq \deg(f)^2$.

Huang (2019)'s Roadmap

Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

- *For any induced subgraph H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, we have $\Gamma(H) \geq h(n)$.*
- *For any Boolean function f, we have $s(f) \geq h(\deg(f))$.*

Theorem (Huang 2019)

For $n \in \mathbb{N}$, if H is a $(2^{n-1}+1)$-vertex induced subgraph of $\mathbb{B}^n$, then

$$\Delta(H) \geq \sqrt{n}.$$

- For any induced subgrpah H of $\mathbb{B}^n$ with $|V(H)| \neq 2^{n-1}$, one of H and $\mathbb{B}^n \setminus H$ contains at least $2^{n-1}+1$ vertices.
- Gotsman and Linial (1992): $s(f) \geq \sqrt{\deg(f)}$.
- Tal (2013): $bs(f) \leq \deg(f)^2$. $\implies$ $bs(f) \leq s(f)^4$.

Idea 1: Largest Eigenvalue

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1, 0, 1\})$ be symmetric with $A_{ij} = 0$ if i, j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1, 0, 1\})$ be symmetric with $A_{ij} = 0$ if i, j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i,j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i,j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j|$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i, j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j| = \left|(\lambda_1 \vec{v})_j\right|$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1, 0, 1\})$ be symmetric with $A_{ij} = 0$ if i, j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j| = \left|(\lambda_1 \vec{v})_j\right| = \left|(A\vec{v})_j\right|$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i,j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j| = \left|(\lambda_1 \vec{v})_j\right| = \left|(A\vec{v})_j\right| = \left|\sum_{i=1}^{m} A_{ji} v_i\right|$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i,j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j| = \left|(\lambda_1 \vec{v})_j\right| = \left|(A\vec{v})_j\right| = \left|\sum_{i=1}^{m} A_{ji} v_i\right|$$

$$\leq \sum_{i=1}^{m} |A_{ji}| \cdot |v_i|$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i, j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j| = \left|(\lambda_1 \vec{v})_j\right| = \left|(A\vec{v})_j\right| = \left|\sum_{i=1}^{m} A_{ji} v_i\right|$$

$$\leq \sum_{i=1}^{m} |A_{ji}| \cdot |v_i| \leq \sum_{i=1}^{m} |A_{ji}| \cdot |v_i|$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i,j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j| = \left|(\lambda_1 \vec{v})_j\right| = \left|(A\vec{v})_j\right| = \left|\sum_{i=1}^{m} A_{ji} v_i\right|$$

$$\leq \sum_{i=1}^{m} |A_{ji}| \cdot |v_i| \leq \sum_{i=1}^{m} |A_{ji}| \cdot |v_i| \leq \Delta(G) \, |v_j| \, .$$

Idea 1: Largest Eigenvalue

Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i,j are not adjacent in G. Then

$$\Delta(G) \geq \lambda_1(A).$$

Proof.

- Let $(\lambda_1, \vec{v})$ be an eigenpair of A. Then $\lambda_1 \vec{v} = A\vec{v}$.
- Let v_j be the component of $\vec{v}$ with the largest absolute value.

$$|\lambda_1| \cdot |v_j| = |\lambda_1 v_j| = \left|(\lambda_1 \vec{v})_j\right| = \left|(A\vec{v})_j\right| = \left|\sum_{i=1}^{m} A_{ji} v_i\right|$$
$$\leq \sum_{i=1}^{m} |A_{ji}| \cdot |v_i| \leq \sum_{i=1}^{m} |A_{ji}| \cdot |v_i| \leq \Delta(G)\,|v_j|\,.$$

- Hence $|\lambda_1| \leq \Delta(G)$.

□

Idea 2: Eigenvalue Interlace

Idea 2: Eigenvalue Interlace

Theorem (Cauchy's Interlace Theorem)

Let $A \in \mathcal{M}_m(\mathbb{R})$ be symmetric. Let B be a $k \times k$ principal submatrix of A for some $m < n$. Then for all $i \in [m]$,

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{i+n-m}(A).$$

Idea 2: Eigenvalue Interlace

Theorem (Cauchy's Interlace Theorem)

Let $A \in \mathcal{M}_m(\mathbb{R})$ be symmetric. Let B be a $k \times k$ principal submatrix of A for some $m < n$. Then for all $i \in [m]$,

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{i+n-m}(A).$$

- If A be a signed adjacency matrix of $\mathbb{B}^n$, and if H an induced subgraph of $\mathbb{B}^n$ with $|V(H)| = 2^{n-1} + 1$, then

$$\lambda_1\left(A_H\right) \geq \lambda_{1+2^n-(2^{n-1}+1)}(A) = \lambda_{2^{n-1}}(A),$$

  where A_H is the corresponding principal submatrix of A.

Idea 2: Eigenvalue Interlace

Theorem (Cauchy's Interlace Theorem)

Let $A \in \mathcal{M}_m(\mathbb{R})$ be symmetric. Let B be a $k \times k$ principal submatrix of A for some $m < n$. Then for all $i \in [m]$,

$$\lambda_i(A) \geq \lambda_i(B) \geq \lambda_{i+n-m}(A).$$

- If A be a signed adjacency matrix of $\mathbb{B}^n$, and if H an induced subgraph of $\mathbb{B}^n$ with $|V(H)| = 2^{n-1} + 1$, then

$$\lambda_1(A_H) \geq \lambda_{1+2^n-(2^{n-1}+1)}(A) = \lambda_{2^{n-1}}(A),$$

  where A_H is the corresponding principal submatrix of A.
- **Magic!** Find a signed adjacency matrix A of $\mathbb{B}^n$ with

$$\lambda_{2^{n-1}}(A) = \sqrt{n}.$$

Idea 3: Magic Signed Adjacency Matrix

[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

Idea 3: Magic Signed Adjacency Matrix

Lemma

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1}, and $-\sqrt{n}$ of multiplicity 2^{n-1}.

[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

Idea 3: Magic Signed Adjacency Matrix

Lemma

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ *whose eigenvalues are* $\sqrt{n}$ *of multiplicity* 2^{n-1}*, and* $-\sqrt{n}$ *of multiplicity* 2^{n-1}*.*

[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

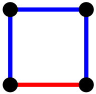
Idea 3: Magic Signed Adjacency Matrix

Lemma

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ *whose eigenvalues are* $\sqrt{n}$ *of multiplicity* 2^{n-1}*, and* $-\sqrt{n}$ *of multiplicity* 2^{n-1}.



[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

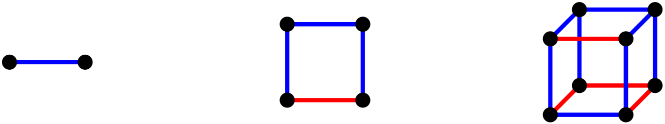
Idea 3: Magic Signed Adjacency Matrix

Lemma

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1}, and $-\sqrt{n}$ of multiplicity 2^{n-1}.



[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

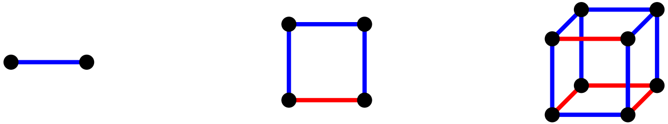
Idea 3: Magic Signed Adjacency Matrix

Lemma

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1}, and $-\sqrt{n}$ of multiplicity 2^{n-1}.



Uses Hadamard's inequality. See Huang's talk at Simons Institute.[3]

---

[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

Idea 3: Magic Signed Adjacency Matrix

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Idea 3: Magic Signed Adjacency Matrix

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Proof.

- Prove by induction that $A_n^2 = nI$.

Idea 3: Magic Signed Adjacency Matrix

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Proof.

- Prove by induction that $A_n^2 = nI$. For $n = 1$, $A_1^2 = I$.

Idea 3: Magic Signed Adjacency Matrix

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Proof.

- Prove by induction that $A_n^2 = nI$. For $n = 1$, $A_1^2 = I$.
- Suppose $A_{n-1}^2 = (n-1)I$. Then

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = \begin{bmatrix} nI & 0 \\ 0 & nI \end{bmatrix} = nI.$$

Idea 3: Magic Signed Adjacency Matrix

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Proof.

- Prove by induction that $A_n^2 = nI$. For $n = 1$, $A_1^2 = I$.
- Suppose $A_{n-1}^2 = (n-1)I$. Then

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = \begin{bmatrix} nI & 0 \\ 0 & nI \end{bmatrix} = nI.$$

- Hence, the eigenvalues of A_n are either $\sqrt{n}$ or $-\sqrt{n}$.

Idea 3: Magic Signed Adjacency Matrix

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Proof.

- Prove by induction that $A_n^2 = nI$. For $n = 1$, $A_1^2 = I$.
- Suppose $A_{n-1}^2 = (n-1)I$. Then

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = \begin{bmatrix} nI & 0 \\ 0 & nI \end{bmatrix} = nI.$$

- Hence, the eigenvalues of A_n are either $\sqrt{n}$ or $-\sqrt{n}$.
- Since $\sum_{\lambda \text{ eigenvalue of } A_n} \lambda = \text{tr}(A_n) = 0$, then exactly half of the eigenvalues of A_n are $\sqrt{n}$, and the rest are $-\sqrt{n}$.

□

Open Questions

Open Questions

- Let G be a "nice" graph with high symmetry. Denote by $\alpha(G)$ the independence number of G, i.e., the size of the largest independent vertex set. Let $f(G)$ be the minimum $\Delta(H)$ over $(\alpha(G)+1)$-vertex induced subgraphs H of G vertices. What can we say about $f(G)$? For which graphs, Huang's method would provide a tight bound?

Open Questions

- Let G be a "nice" graph with high symmetry. Denote by $\alpha(G)$ the independence number of G, i.e., the size of the largest independent vertex set. Let $f(G)$ be the minimum $\Delta(H)$ over $(\alpha(G)+1)$-vertex induced subgraphs H of G vertices. What can we say about $f(G)$? For which graphs, Huang's method would provide a tight bound?
- Let $g(n,k)$ be the minimum t such that every t-vertex induced subgraphs H of $\mathbb{B}^n$ has $\Delta(H) \geq k$. Huang (2019) shows $g(n, \sqrt{n}) = 2^{n-1} + 1$.

Open Questions

- Let G be a "nice" graph with high symmetry. Denote by $\alpha(G)$ the independence number of G, i.e., the size of the largest independent vertex set. Let $f(G)$ be the minimum $\Delta(H)$ over $(\alpha(G)+1)$-vertex induced subgraphs H of G vertices. What can we say about $f(G)$? For which graphs, Huang's method would provide a tight bound?
- Let $g(n,k)$ be the minimum t such that every t-vertex induced subgraphs H of $\mathbb{B}^n$ has $\Delta(H) \geq k$. Huang (2019) shows $g(n,\sqrt{n}) = 2^{n-1}+1$.
- The best separation between block sensitivity and sensitivity is $bs(f) = \frac{2}{3}s(f)^2 - \frac{1}{3}s(f)$ (Ambainis and Sun 2011). Close the gap between this and the quartic upper bound.

Huang's Timeline

Nov 2012: I was introduced to this problem by Michael Saks when I was a postdoc at the IAS, and got immediately attracted by the induced subgraph reformulation. And of course, in the next few weeks, I exhausted all the combinatorial techinques that I am aware of, yet I could not even improve the constant factor from the Chung-Furedi-Graham-Seymour paper, or give an alternative proof without using the isoperimetric inequality.

Around mid-year 2013: I started to believe that the maximum eigenvalue is a better parameter to look at, actually it is polynomially related to the max degree, i.e \sqrt{\Delta(G)} \le \lambda(G) \le \Delta(G). And in some sense it reflects some kind of "average degree" (unfortunately the average degree itself could be very small, something like \sqrt{n}/2^n).

2013-2018: I revisited this conjecture every time when I learn a new tool, without any success though. But at least thinking about it helps me quickly fall asleep many nights.

Late 2018: After working on a project (with Pohoata and Klurman) that uses Cvetkovic's inertia bound to re-prove Kleitman's isodiametric theorem (it is another cute proof using algebra solving extremal combinatoiral problems), and several semesters of teaching a graduate combinatorics course, I started to have a better understanding of eigenvalue interlacing, and believe that it might help this problem. For example, applying interlacing to the original adjacency matrix, one can already show that with (1/2+c) proportion of vertices, the induced subgraph has maximum degree c'*\sqrt{n}. I don't think this statement could follow easily from combinatorial arguments. Yet at that time, I was hoping for developing something more general using the eigenspace decomposition of the adjacency matrix, like in this unanswered MO question:

https://mathoverflow.net/questions/331825/generalization-of-cauchys-eigenvalue-interlacing-theorem

June 2019: In a Madrid hotel when I was painfully writing a proposal and trying to make the approaches sound more convincing, I finally realized that the maximum eigenvalue of any pseudo-adjacency matrix of a graph provides lower bound on the maximum degree. The rest is just a bit of trial-and-error and linear algebra.

Thank you!